A Nonlinear Progress to Modern Soliton Theory

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By

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PREFACE

This account sets out with the oft-quoted observations by the celebrated engineer John Scott Russell in 1834 of the singular wave of translation and of his subsequent experiments to reproduce this remarkable solitonic phenomenon. In terms of geometry, the classical work of Bour, Bäcklund and Bianchi and its consequences are recounted, notably with regard to the nonlinear superposition principles known as permutability theorems which later were shown to be generic to soliton systems and allow the analytic description of multi-soliton interaction. Here, the key connections between privileged motions of certain curves and diverse solitonic surfaces so generated are described.

Modern soliton theory, since its inception, has been the subject of intense research with well-documented diverse applications in physics. However, remarkable connections between soliton theory and nonlinear continuum mechanics via application of Bäcklund transformations with origin in classical geometry have had a separate development. These have not previously been described in a single account. Here, however, it is recorded how seminal papers by the distinguished mathematician Karl Loewner in 1950 and 1952 on the application of Bäcklund transformations in gasdynamics have *mutatis mutandis*, in recent times, been shown to have wide application in nonlinear continuum mechanics notably in such disparate areas as elastostatics, elastodynamics, superelasticity, magneto-hydrostatics and magneto-gasdynamics. Moreover, a due re-interpretation as extension of Bäcklund transformations of Loewner-type underlies the construction of a novel master system in 2+1-dimensional soliton theory. Intriguing connections which exist between the infinitesimal Bäcklund transformations of Loewner and certain aspects of relativity are recorded.

The aim of the present monograph is to provide a detailed account of key analytic developments and diverse physical applications in the nonlinear progress to modern soliton theory. The work may be regarded as, in part, a supplement to the mathematical accounts:

C. Rogers and W.F. Shadwick, Bäcklund Transformations and Their Applications, Academic Press, Mathematics in Science and Engineering Series, New York (1982)

C. Rogers and W.K. Schief, Bäcklund and Darboux Transformations. Geometry and Modern Applications in Soliton Theory, Cambridge Texts in Applied Mathematics, Cambridge University Press (2002)

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The author wishes to acknowledge the contributions to the subject of soliton theory made by his distinguished colleagues Professor Wolfgang Schief of the University of New South Wales and Professor Boris Konopelchenko of the Università del Salento without whom this monograph, as it stands, could not have been written.

In addition, deep appreciation is due to Professor Athanassios Fokas, of the University of Cambridge for provision of an appendix on diverse aspects of the inverse scattering method and its application in soliton theory.

Chapter I

Historical Introduction

The accepted advent of modern soliton theory with its genesis in observations originally made by the engineer John Scott-Russell in 1834 came with the computer analysis in 1965 by Zabusky and Kruskal [1] of a lattice system which, in a continuum limit, resulted in the so-called KdV equation. The latter had been derived *ab initio* in 1895 by two Dutch mathematicians Korteweg and de Vries [2] to describe the propagation of long waves in a rectangular channel. However, it should be recorded that this canonical equation had been earlier obtained by Boussinesq [3] in 1871 and set down in a subsequent extensive *mémoire* [4] published in 1877. The mathematical model which resulted in the KdV equation importantly, served to validate the empirical observations of Scott-Russell in connection with a singular wave propagation phenomenon arising out of the motion of barges on a canal. Thus, the large amplitude solitary wave so observed propagated as an isolated hump which remained above the undisturbed level of the canal and which travelled with a speed proportional to its height. In a subsequent report to the British Association in 1844, Scott-Russell [5] described novel experiments he had conducted which reproduced the type of solitary wave he had observed ten years earlier on an Edinburgh canal. However, the restricted length of the water tank in which the experiments were conducted precluded the discovery of the remarkable nonlinear interaction properties of two or more such waves. This had to await a computer study by Zabusky and Kruskal [1] on aspects of the celebrated Fermi-Pasta-Ulam problem [6]. The KdV equation arises out of this work as a continuum limit of an associated anharmonic lattice model. The singular solitary waves that were to be termed *solitons* in [1] have the remarkable property that, subsequent to a complicated interaction, they emerge asymptotically with unchanged speed, amplitude and shape. However, the analytical description of this novel nonlinear interaction process at that stage remained unknown.

It was a crucial connection with a classical geometric study of the transformation of pseudospherical surfaces originally published by

Bäcklund [7] in 1883 which ultimately led to a generic mathematical method for the description of the nonlinear interaction of solitons in the diverse physical contexts in which they were subsequently shown to occur [8]. It had been Edmund Bour [9] in 1862 who originally derived via the Gauss-Mainardi-Codazzi system for pseudospherical surfaces another nonlinear equation later shown to be canonical in modern soliton theory. This is what was to be later termed the sine Gordon equation. It was subsequently re-derived independently by the distinguished geometers Bonnet [10] in 1867 and Enneper [11] in 1868. It was later established by Bäcklund in [7] that the sine Gordon equation derived by Bour admits invariance under a novel class of transformations containing a key Lie-group parameter. In a masterly advance in [12] by the Italian mathematician Bianchi in 1892, it was established that a commutative property possessed by Bäcklund transformations with distinct such Lie parameters leads to a novel nonlinear superposition principle known as a permutability theorem whereby a chain of pseudospherical surfaces may be generated via the iterated action of a purely algebraic procedure. That the permutability theorem of Bianchi was to have important application in nonlinear physics had to await the seminal work of Seeger *et al* [13] published in 1953 on the subject of crystal dislocations. Therein, within the framework of Frenkel and Kontorova's dislocation theory [14], Bianchi's permutability theorem was adduced to obtain an analytic description of the nonlinear interaction of so-called *eigenmotions* later to be called breathers and kinks in the terminology of modern soliton theory. In this remarkable paper the characteristic solitonic properties to be subsequently uncovered in the computer study in 1965 for the KdV equation, namely, preservation of speed and shape together with a concomitant phase shift subsequent upon nonlinear interaction were all derived via the application of Bianchi's classical permutability theorem.

The trinity of canonical soliton equations that emerged in the early period of modern soliton theory was completed by what has become known as the nonlinear Schrödinger (NLS) equation. This had its roots in a classical hydrodynamic analysis by Da Rios [15] in 1906 of the motion of a thin isolated vortex filament travelling without stretching in an unbounded incompressible fluid. A coupled nonlinear system for the temporal evolution of the curvature κ and torsion τ of the vortex filament was derived therein. Hasimoto [16] subsequently in 1972 was to demonstrate that this Da Rios system may be encapsulated in the nonlinear Schrödinger equation later to be shown to be ubiquitous in soliton theory. That the NLS equation admits an auto-Bäcklund transformation which may be constructed by purely geometric means was to be established in [17].

The present account starts with the discovery of the solitary wave phenomenon by Scott-Russell in 1834 and traces subsequent diverse theoretical developments and applications which subsequently emerged in modern soliton theory. Importantly, the key roles of serendipity, analogy and connection become apparent in turn in this nonlinear progress.

CHAPTER II

SCOTT-RUSSELL'S CHANCE ENCOUNTER

In August 1834, a naval architect and engineer, John Scott Russell was riding his horse along the banks of what is believed to be the Union Canal on the outskirts of Edinburgh. This passes below the present day campus of Heriot-Watt University. He was to observe a remarkable phenomenon that was destined to have far-reaching scientific consequences that extend to the present day. The details were subsequently reported by Scott-Russell in [5]. Therein, in what is now a celebrated quotation, the initial circumstances of his discovery were related as follows:

'I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses when the boat suddenly stopped-not so the mass of water in the channel which it had set in motion: it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a solitary elevation, a rounded smooth and well-defined heap of water, which continued its course along the canal apparently without change of form or diminution of speed. I followed it on horseback and overtook it still rolling at some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance encounter with the singular and beautiful phenomenon which I have called the Wave of Translation, a name which it now generally bears: which I have since found to be an important element in almost every case of fluid resistance, and ascertained to be of the type of that great moving elevation of the sea, which, with the regularity of a planet, ascends our rivers and rolls along our shores.'

Subsequent to its discovery, Scott-Russell adopted the term *Large Solitary Wave* to describe the phenomenon. Later investigators such as Sir George Stokes and Lord Rayleigh likewise used this terminology and this became the common practice.

Scott-Russell's 'chance encounter' subsequently motivated the Union Canal Company to pay for the boats which were constructed in 1835 for the experiments upon which he later reported. The Encyclopaedia Britannica of 1886 described his involvement thus:

'Having been consulted as to the possibility of applying steam-navigation to the Edinburgh and Glasgow canal, he replied that the question could not be answered without experiment, and that he was willing to undertake such if a portion of the canal were placed at his disposal.'

On the 100th anniversary of Scott-Russell's death in 1982, an attempt was made on the Union Canal near Edinburgh to reproduce a solitary wave of the type he originally observed in 1834. A group of leading international specialists in the by then well-established area of modern soliton theory was assembled and an appropriate boat provided for the purpose. A horseman resplendent in early Victorian garb was held in readiness on the tow path alongside the canal and prepared to give chase should the occasion arise. In the event, he was never called into action. The external circumstances attending the experiment had possibly not been optimum. Thus, a large hospitality tent had been erected near the canal bank and many of the senior academics had early succumbed to its attractions. As a result, they were unable to contribute, at least in a helpful manner, to the conduct of the experiment.

Indeed, the authenticity of the solitary wave phenomenon as observed by Scott-Russell in 1834 had been early called in question. A controversy ensued which was to involve some of the most eminent men of Victorian science, in the main arrayed against Scott-Russell. The matter was not finally resolved in his favour until 1876 when Lord Rayleigh[1] correctly predicted the shape of Scott-Russell's solitary wave mathematically in a paper published in the Philosophical Magazine [18]. However, this work did not enjoy wide currency and the matter was only at last set to rest with the paper of Korteweg and de Vries in [2]. The mathematical model resulting in what is now known as the KdV equation produced results consistent with the original solitary wave observations of Scott-Russell.

[1] Lord Rayleigh (J.W. Strutt). Nobel Prize in Physics 1904, President of the Royal Society 1905-1908

John Scott-Russell

CHAPTER III

SOLITON INTERACTION

There was a key property of the type of solitary wave observed by Scott-Russell that was to be the root of its subsequent importance in diverse areas of physics and nonlinear continuum mechanics. This relates to their remarkable interaction properties. These were revealed in the computer calculations of Zabusky and Kruskal in [1] albeit in another physical context. This concerned an apparently totally unrelated computer study of oscillations in certain coupled systems involving connected masses and springs—the so-called Fermi-Pasta-Ulam problem [6]. In an inspired interpretation of their computer output, recurrence phenomena were detected which is now associated with soliton survival following interaction. Moreover, the discrete mathematical model of their mass-spring system, or lattice as it is commonly called, reduces in a continuum limit to none other than the KdV equation which seventy years earlier had served to validate John Scott-Russell's observations concerning the singular waves of translation as recorded in 1834.

The analytic description of the complicated nonlinear interaction process that occurs when solitons as described by the KdV equation meet had to await the work of Wahlquist and Estabrook [19] in 1973 who applied a nonlinear superposition principle generated via a Bäcklund transformation [20]. The hierarchy of solutions of the KdV equation as constructed by the iterated action of the nonlinear superposition principle (permutability theorem) on the seed vacuum solution correspond to multi-soliton pulses. The latter, following interaction, have the extraordinary property that they asymptotically regain their shape and speed subsequent to nonlinear interaction, emerging with no more than a phase shift on position.

The fact that the KdV equation serves as a common mathematical model in apparently totally unrelated areas attests to the crucial rôle that applied mathematics plays in revealing hidden links between disparate physical phenomena. Thus, in particular, the KdV equation had been re-derived in 1960 by Gardner and Morikawa in an analysis of the propagation of hydromagnetic waves [21]. It subsequently has been shown to model a diversity of solitonic wave phenomena in the theory of solids, liquids, gases and plasmas. In each case, the remarkable characteristic properties of multi-soliton interaction are admitted since they are a natural consequence of the common underlying KdV model. In particular, the generic collision properties must carry over to the interaction of the type of singular of translation as originally observed by Scott-Russell. Indeed, there is evidence in his report to the British Association in 1844 that he was aware of the properties of two-soliton interaction and evolution. In this connection, there was a sketch presented of a so-called *double wave* that, in Scott-Russell's words

> 'represents the genesis of a large low column of fluid of a compound or double wave of the first order, which immediately breaks down by spontaneous analysis into two, the greater moving faster and altogether leaving the smaller'.

A seminal contribution to the genesis of modern soliton theory resides in the work of Seeger *et al* [13] in their application of the classical Bäcklund transformation with its associated permutability theorem in the context of crystal dislocation theory. Thus, at the United Aircraft Company in the USA in the mid 1960s, a research physicist called George Lamb was investigating certain aspects of pulse propagation in lasers. In so doing, he was led to derive as a mathematical model none other than the canonical sine-Gordon equation descriptive in asymptotic co-ordinates in classical geometry of surfaces of constant negative Gaussian curvature, namely pseudospherical surfaces [22]. Lamb was aware of the results of Seeger *et al* in [13] and accordingly sought to apply Bianchi's permutability theorem in the same manner to model the nonlinear interaction properties of ultrashort light pulses. Novel solitonic decomposition properties of such pulses were thereby uncovered by Lamb [23] and subsequently Barnard [24] via iterative application of the Bianchi permutability theorem. A detailed account of the procedure is presented in [20]. Experimental evidence for this phenomenon was reported in the work of Gibbs and Slusher [25] in 1970 concerning the observed decomposition of a 6π pulse into three 2π pulses in *Rb* vapour.

The physical applications of the Bäcklund transformation of classical geometry derived in this period were extended to superconductivity by Scott [26]. Thus, work by Josephson [2] [27] celebrated for his pioneering contributions in quantum tunnelling and superconductivity led in [26] to a particular reduction to a sine-Gordon equation. Propagation along a one-dimensional Josephson tunnel junction was investigated therein with this

[2] Brian Josephson, Nobel Prize in Physics 1973

model and the classical Bäcklund transformation was applied to derive novel conservation laws for pulses with multi-flux quanta.

The nonlinear Schrödinger equation with its origin in the equivalent geometric Da Rios system derived in a hydrodynamic context has subsequently been shown to model diverse physical phenomena with underlying soliton structure. In 1965, Kelley [28] derived *ab initio* a version of the NLS equation in connection with pulse propagation in nonlinear optics. An equivalent result was obtained independently in the same context and year by Talanov [29]. Later in 1968, Taniuti and Washimi [30] derived the NLS equation in connection with the self-trapping of hydromagnetic waves in a cold plasma. In contemporary work, Asano *et al* [31] obtained this canonical solitonic equation in a study of the modulation of nonlinear waves with application to the analysis of the propagation of electron plasma pulses. In the same period, Zakharov [32] derived the NLS equation in an analysis of deep-water gravity waves. It has since been shown to be ubiquitous in nonlinear physics. In particular, remarkably, the classical Korteweg capillarity system originally set down in 1901 admits an integrable NLS reduction (Antanovskii *et al* [33]).

The auto-Bäcklund transformation for the canonical integrable nonlinear Schrödinger equation had been originally obtained by Lamb [34] in 1977 in connection with privileged spatial motions of certain curves and later via the original Da Rios system and application of a geometric formulation previously adopted in magneto-gasdynamics [35]. The importance of the Da Rios system of 1906 had been recognised by the distinguished Italian mathematian Levi-Civita and it was reported by him in a survey later published in [36].

Thus it was that in independent work conducted in the nineteenth century by the engineer John Scott-Russell and the trio of mathematicians Bäcklund, Lie and Bianchi was brought together in the mid twentieth century to describe analytically the remarkable phenomenon of solitonic nonlinear interaction. The theory and diverse applications of soliton theory in physics are well-documented in such works as [37, 38]. In addition, the classical geometry underlying such systems is covered in detail in [8]. However, it has subsequently emerged that aspects of soliton theory arise in disparate areas of nonlinear continuum mechanics such as, *inter alia*, nonlinear elastostatics and elastodynamics, superelasticity, the geometry of membranes with application in liquid crystal theory, the deformation of fiber-reinforced materials, magneto-hydrostatics, magneto-gasdynamics and capillarity theory. To date however, there is no text such as the present that brings a description of these applications of soliton theory together in a single account. Here, in particular, the diverse recent applications in nonlinear continuum mechanics are described which arise in an unexpected manner from re-interpretation and extension of work contained in two papers by Karl Loewner published in 1950 and 1952 on certain applications of Bäcklund transformations in classical gasdynamics. However, at this juncture, a return is made to John Scott-Russell who set in motion this remarkable progress with his chance encounter in 1834 with that celebrated wave of translation.

CHAPTER IV

JOHN SCOTT-RUSSELL: A GREAT VICTORIAN ENGINEER AND NAVAL ARCHITECT

Such was the title of the biography of Scott-Russell published in 1977 by the engineer George Emmerson, who is perhaps best known for his social history of Scottish dance entertainingly titled

> 'Rantin Pipe and Tremblin String'.

However, there is no mention in the biography of Scott-Russell's discovery of the wave of translation and the associated development of modern soliton theory. Rather, emphasis is placed, in particular, on Scott-Russell's collaboration with Brunel in the design of ocean-going ships such as the Great Eastern. Indeed, one of the most striking photographs in Emmerson's biography depicts a scene at the launch of the Great Eastern. This historic ship was of gigantic proportions and was not to be equalled in size for almost 50 years. Therein, in characteristic pose complete with the inevitable cigar is the expansive Isambard Kingdom Brunel. In the same picture, but somewhat separated stands John Scott-Russell, Brunel's engineering partner on the project, the naval architect of the great ship. At the age of 49 he was at the height of his powers. We continue in Emmerson's words:

> 'Yet there fell a shadow on his expectations and deserts, a strange antipathy of fortune hinted at by his division from the others in this historical photograph'.

This is not the place to recount the tempestuous passage that was Scott-Russell's particular nonlinear progress through life. This is vividly recounted in Emmerson's estimable book. Thus, we shall not dwell upon Russell's rôle as Royal Commissioner for the Great Exhibition of 1851, as naval architect and designer of the so called *wave line* ships and his involvement as a peace negotiator in the American Civil War. Nor shall we expand upon his independent discovery of the Doppler Effect and his calculation of the depth of the Earth's atmosphere. Rather, we shall touch upon isolated aspects of Scott-Russell's career which bear upon our present topic and serve, to a degree, to complement Emmerson's work.

John Russell was born in 1808 in an eighteenth century weavers' cottage in the environs of Glasgow. He was later to add to his name that of his mother, neé Agnes Clark Scott. It was through her also that came the only elevated aspect of his inheritance, namely possession of a silver teapot from which tea had been poured for Dr. Johnson during the celebrated Highland jaunt of 1773 with his amanuensis James Boswell.

Scott-Russell entered upon his university studies at St. Andrews and proceeded to the University of Glasgow where he graduated as an MA in 1825 at a mere 17 years of age. This had been the Glasgow of Adam Smith and the great 'mechanic' James Watt. It was some twenty years later to see Robert Napier established as one of the world's greatest marine engineers and ship builders. The deep local commitment to the harnessing of the novel steam engines of that period to commercial ends evidently made a deep imprint on the young Scott-Russell as his later career was to show.

However, it was to the works of eighteenth century classical applied mathematicians that Scott-Russell was to owe his greatest debt—as he was later to bear eloquent testament. Thus, at University he was exposed to the works of Euler as well as to Lagrange's *Mécanique Analytique*, the Marquis de Laplace's *Traité de Mécanique Celeste* and, importantly, Poisson's *Mémoire sur La Théorie des Ondes*. To these he added knowledge of Young's *Theory of Elasticity* as well as the more arcane Brewster's *Theory Mechanical Philosophy of Dr. Robinson*. Some 40 years on, in 1865 Russell was to record their influence on him thus:

> 'I drew in their spirit from their books, into which, it seemed to me they had breathed their souls. If I now know anything, it is because I see it with their eyes, and search into it with their way of asking, and put it into their way of thinking'.

It was accordingly entirely natural that Scott-Russell should choose to set out as a teacher of mathematics, this in Edinburgh in 1825. However, when his academic duties allowed him, in the next decade he pursued an independent interest in the design of steam contraptions. This work became well-known locally and such were its commercial possibilities that in 1834 a group of Edinburgh business men formed the Scottish Steam Carriage Company and provided the capital for the production of a prototype steam-carriage to be designed by Scott-Russell. However, it was then that fate was to take a hand: Scott-Russell was offered the contract by the Union Canal Company to which we have already alluded. Through this chance happening the soliton was both discovered and later reproduced experimentally.

Let us recount in Scott-Russell's words his experimental reproduction of soliton propagation. His object in the experiment was to discover what became of a heap of water added to a body of water at rest. To this end, he constructed a small trough of about one foot square in cross-section and of great length, and filled it with water to a depth of six inches. This, in Scott-Russell's own words, is what happened:

> 'I made a little reservoir of water at the end of the trough, and filled this with a little heap of water, raised above the surface of the fluid in the trough. The reservoir was fitted with a movable side or partition; on removing which, the water within the reservoir was released. It will be supposed by some that on removal of the partition the little heap of water settled itself down in some way in the end of the trough beneath it, and that this end of the trough became fuller than the other, thereby producing an inclination of the water's surface, which gradually subsided until the whole got level again. No such thing. The little released heap of water acquired life, and commenced a performance of its own, presenting one of the most beautiful phenomena I ever saw. The heap of water took a beautiful shape of its own, and instead of stopping, ran along the whole length of the channel as quiet and as much at rest as it had been before. If the end of the channel had just been so low that it could have jumped over, it would have leaped out, disappeared from the trough, and left the whole canal at rest just as it was before. This is a most beautiful and extraordinary phenomenon, the first day I saw it was the happiest day of my life. Nobody had even the good fortune to see it before or, at all events, to know what it meant. It is now known as the *solitary wave of translation*. No-one before had fancied a solitary wave to be a possible thing. When I described this to Sir John Herschel, he said, "It is merely half of a common wave that has been cut off". But it is not so, because the common waves go partly above and partly below the surface level: and not only that, but its shape is different. Instead of being half a wave, it is clearly a whole wave, with this difference, that the whole wave is not above and below the surface alternatively, but always above it. So much for what a heap of water does: it does not stay where it is, but travels to a distance'.

Scott-Russell added other key comments to the above as follows:

> 'The second fact which I have ascertained in reference to this solitary wave is a curious one, viz: it will carry the water to a distance almost incredible. I have followed such a wave, on horseback and

> by other means, for miles; and I have met such a wave, of moderate size after it had gone five or six miles.....
>
> The next important fact that I have discovered is this: that whenever you force the bow of a ship through the water, you produce such a wave as I have described; and this is the travelling or carrier wave..... The bow of the ship raises a little heap of water before it, and that heap, once raised, runs away.....'

By 1842 at the age of 34, such were Scott-Russell's scientific credentials that he was a strong candidate for the Chair of Mathematics at Edinburgh University vacated by the eminent Professor Wallace. In support of his candidature, Sir William Hamilton, the Astronomer Royal of Ireland spoke of Scott-Russell as 'a person of active and inventive genius' while Sir David Brewster drew attention to '(Scott-Russell's) peculiar facility for expressing and explaining his ideas, and the fluency and eloquence with which he communicated an account of his hydrodynamic researches to the British Association which excited the admiration of a numerous and distinguished audience.'

It is a matter of curiosity that some 140 years later it emerged that some of Sir William Hamilton's most important mathematical research has intimate connections with soliton theory. These are described in the authorative monograph of Faddeev and Takhtajan [38].

Despite his pre-eminent suitability, Scott-Russell was not appointed to the Chair at Edinburgh. Nor indeed was he to ever hold a university position. Thus, he was led to a career in shipbuilding and commerce. Despite this, his academic standing was such that he was elected an FRS in 1849.

Scott-Russell's interest in hydrodynamics did not die with his academic ambitions. In particular, in the 1860's he seemed to have anticipated the revolutionary idea of boundary layer to be postulated by Prandtl [39] some 40 years later. Thus, in connection with a study of the resistance to the passage of a ship through water, Scott-Russell drew attention to a 'a ribbon of water' the innermost layer sticking to the ship and moving on with it.

Scott-Russell died in 1882 and his treature *The Wave of Translation in the Oceans of Water, Air and Ether* was published posthumously in 1885.

Controversy and Vindication

Controversy had attended the concept of the great wave of translation, starting with the doubt cast upon both the novelty and nature of Scott-Russell's observations by Sir John Herschel. In 1845, Sir George Biddell Airy, the Astronomer Royal stated in his treatise *Tides and Waves* that 'we are not disposed to recognise this wave as deserving the epithets *great* or *primary*...'.

Sir George Stokes in the British Association Report of 1846 on the other hand was to say 'it is the opinion of Mr Russell that the solitary wave is a phenomenon *sui genesis* in nowise deriving its character from the circumstances of the generation of the wave. His experiments seem to render this conclusion probable. Should it be correct the analytical character of the solitary wave remains to be discovered'.

In the event, it was not to be until 1876 that Lord Rayleigh [18] in work contemporary with that by Boussinesq was to resolve the controversy analytically. Not only did he calculate mathematically the shape of the wave of translation but, remarkably, he derived the same expression for speed of propagation that had been obtained empirically by Scott-Russell 40 years earlier. The dependence of the speed of propagation of the solitary wave upon its amplitude is made precise by this mathematical expression and accords with empirical observations.

Thus it was that at the end of his days, Scott-Russell was scientifically vindicated although he was never accorded, apart from the FRS, the honours bestowed on some lesser lights by the Establishment. Moreover, the Encyclopaedia Britannica's article of 1886 on Scott-Russell continued to propagate the error that Sir George Airy was correct in his dismissal of Scott-Russell's work.

The results of the mathematical model adopted by Korteweg and de Vries in their classical paper of 1895, together with the previous contributions of Boussinesq and Lord Rayleigh nevertheless placed the matter beyond dispute. The historical work of John Scott Russell was to be the *sine qua non* for the subsequent development of modern soliton theory with its concomitant deep analysis of diverse nonlinear physical phenomena.

CHAPTER V

BÄCKLUND TRANSFORMATIONS AND THE INVERSE SCATTERING METHOD: A BÄCKLUND MEETING

In 1968, Robert Miura in work at Princeton, described a remarkable transformation which links the KdV equation to what has become known as the modified Korteweg (mKdV) equation [40]. The latter had been originally derived by Zabusky [41] in connection with wave propagation in anharmonic lattices. It was also set down in 1969 by Kakutani and Ono [42] in an analysis of the propagation of hydromagnetic waves in a cold collisionless plasma.

The Miura transformation provides a novel connection between the KdV and mKdV equation via a Riccati-type equation. Crucially, linearization of the latter together with application of a Galilean invariance of the KdV equation leads to a linear parameter-dependent Sturm-Liouville reduction which constitutes a Schrödinger-type scattering problem. This had provided the basis for the introduction of the celebrated inverse scattering method by Gardner *et al* [43]. The procedure was later to allow the analytic treatment of wide classes of initial value problems for solitonic equations and has an extensive literature (see e.g. [37, 44, 45] and work cited therein). It may be regarded as an extension to soliton theory of the classical method of Fourier analysis.

Bäcklund, subsequent to his classical paper of 1883 was to continue his study of such invariant transformations contained therein [46]. Clairin [47] in 1902 later introduced a procedure to construct Bäcklund transformations in a systematic manner with subsequent wide applicability in modern soliton theory. This approach was described by Lamb in [34] and used to construct auto-Bäcklund transformations, in turn, for the KdV, mKdV and nonlinear Schrödinger equations. In [20], Clairin's classical procedure was used to establish *ab initio* that, remarkably, the Miura transformation is nothing but the spatial part of a Bäcklund transformation which links the KdV and mKdV equations.

In 1972, Zakharov and Shabat [48] demonstrated via a method previously introduced by Lax [49] that the NLS equation is likewise amenable to the inverse scattering method. Ablowitz *et al* [50, 51] then in a major advance constructed, via compatibility conditions imposed on a linear representation of some generality, a wide range of nonlinear evolution equations all of which are amenable to the inverse scattering method. It is recalled that, in geometric terms, in the classical theory of pseudo-spherical surfaces, it is compatibility of the linear Gauss system in asymptotic co-ordinates which leads to the nonlinear Mainardi-Codazzi system reducible to the sine Gordon equation as originally obtained by Bour in 1862. The AKNS system as set down in [50] incorporates three major classes of nonlinear evolution equations of which the sine Gordon, KdV and nonlinear Schrödinger equations are representative canonical members.

Thus, by 1974 the skeletal theoretical structure of modern soliton theory was in place. Moreover, the central role of Bäcklund transformations both with regard to the generation of multi-solitons and the analytic description of their nonlinear interaction as well as their key role via the Miura transformation in the genesis of the inverse scattering transform had been internationally recognised. Thus, in 1974 under the auspices of the National Science Foundation in the USA, Robert Miura organised what later emerged as an historic meeting at Vanderbilt University entitled [52]:

Bäcklund Transformations, the Inverse Scattering Method, Solitons & Their Applications

There ensued subsequent to this interaction intensive international research into the theory and application of both continuous and discrete solitonic systems which continues to the present day.

Invited Lectures

• K.E. Lonngren	Experiments on Solitary Waves
• F.Y.F. Chu	Stimulated Raman and Brillouin Scattering and the Inverse Method
• R. Hirota	Direct Method of Finding Exact Solutions of Nonlinear Evolution Equations
• G.L. Lamb	Bäcklund Transformations at the Turn of the Century
• A.C. Scott	The Application of Bäcklund Transforms to Physical Problems

• C. Rogers	On Applications of Generalized Bäcklund Transformations to Continuum Mechanics
• F.B. Estabrook	Some Old and New Techniques for the Practical Use of Exterior Differential Forms
• H.D. Wahlquist	Bäcklund Transformation of Potentials of the Korteweg-de Vries Equation and the Interaction of Solitons with Cnoidal Waves
• J.P. Corones & F.J. Testa	Pseudopotentials and Their Applications
• H. Rund	Variational Problems and Bäcklund Transformations associated with the Sine-Gordon and Korteweg-de Vries Equations and their Extensions
• A.C. Newell	The Interrelation between Bäcklund Transformations and the Inverse Scattering Transform
• H.H. Chen	Relation between Bäcklund Transformations and Inverse Scattering Problems
• H. Flaschka & D.W. McLauglin	Some Comments on Bäcklund Transformations, Canonical Transformations, and the Inverse Scattering Method

CHAPTER VI

NONLINEAR CONTINUUM MECHANICS: SOLITONIC CONNECTIONS AND APPLICATION OF BÄCKLUND TRANSFORMATIONS

Haar [53] in 1928 in an adjoint variational context, introduced a novel class of transformations which leaves invariant, up to the pressure-density relation, the governing equations of two-dimensional, irrotational steady gasdynamics. Bateman[3] [54] in a subsequent study of lift and drag aspects of this homentropic system established its invariance under a class of relations that have come to be termed reciprocal transformations. The latter characteristically involve, in particular, key changes of the independent variables with validity based on conservation laws admitted by the system. The reciprocal transformations of Bateman were subsequently discussed by Tsien [55] in connection with the celebrated Kármán-Tsien model pressure-density relation which, in turn, has its roots in a classical work of Chaplygin [56] on gas jets. Thus, in the case of a model Kármán-Tsien law, the reciprocal transformations of Bateman may be used to link, in subsonic flow, the potential two-dimensional gasdynamic system of [54] with an associated tractable hydrodynamic system encapsulated in a Cauchy-Riemann system. It was later established by Bateman [57] that, remarkably, the reciprocal relations of [54] constitute nothing but a particular class of Bäcklund transformations. Reciprocal-type transformations were subsequently shown to have important applications in modern soliton theory. Thus, conjugated with gauge transformations, they link the canonical AKNS and WKI inverse scattering schemes [58] as well as solitonic hierarchies and their constituent members [8].

The application of Bäcklund transformations to the gasdynamics system of [54] was developed in a systematic manner in [59, 60]. Therein, Loewner[4] who is recognised notably, for his contributions to the theory of

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[3] H. Bateman, Trinity College Cambridge FRS 1928

[4] Karl Loewner received his doctorate in 1917 at the then German University of Prague under the supervision of the distinguished mathematician Georg Pick who

continuous groups and seminal analysis of the Bierberbach conjecture, introduced a class of matrix Bäcklund transformations which, on re-interpretation and extension, were to have important application in both nonlinear continuum mechanics and soliton theory. In [59], the concern was with the reduction via Bäcklund transformations of the classical hodograph system of two-dimensional irrotational gasdynamics to appropriate tractable canonical forms in subsonic, transonic and supersonic régimes, namely, the Cauchy-Riemann system, the Tricomi equation and classical linear wave equation respectively. The Bäcklund transformations introduced therein provided a unifying structure for hitherto unrelated work on approximation theories in homentropic irrotational two-dimensional gasdynamics based on model constitutive laws with application to a wide range of boundary value problems [61]. Multi-parameter extensions of the reciprocal-type Bäcklund transformations to rotational gasdynamics were constructed in [62].

The application of Bäcklund transformations of Loewner-type is not however restricted to gasdynamics. Indeed, particular nonlinear model stress-strain laws equivalent *mutatis mutandis* to those derived by Loewner in a gasdynamic context were subsequently and independently derived by Cekirge and Varley [63] and Kazakia and Varley [64] in a Lagrangian analysis of wave propagation in bounded nonlinear elastic media. This nonlinear model law approach was extended to the analysis of the propagation of plane-polarized waves in slabs of nonlinear dielectric material of finite extent by Kazakia and Venkataran [65]. Important connection may be made with a termination of Bergman series approach as adopted subsequently in an analysis of electromagnetic wave propagation and the construction of model **D**-**E** and **B**-**H** associated with canonical tractable reduction in nonlinear dielectric media [66]. The link between the termination of Bergman-type series and the derivation of model constitutive relations via Bäcklund transformations which allows such reduction was originally recorded in the context of impact wave propagation in inhomogeneous visco-elastic media [67]. Therein, termination of a Bergman-type wave front expansion was linked to model viscoelastic parameter relations of Loewner-type which may be derived via canonical reduction of the kind of Bäcklund transformation originally introduced in a gasdynamic context by Loewner in [59]. The role of such an alternative termination of Bergman series approach in classical inhomogeneous elastostatics was detailed in [68]. In [69,70] application was made to solve classes of anti-plane crack boundary value problems in elastic media with variable shear modulus. In

had chaired the committee which appointed Albert Einstein to the Chair of Mathematical Physics there in 1911