Mathematics for Intermediate Teachers

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From Models to Methods

By

Ann Kajander

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I was greatly privileged to have heard Seymour Papert speak on two occasions during my earlier years as a mathematics educator, and this inspiration has left an indelible mark on my beliefs about the emotional satisfaction of mathematical play, exploration, and deep understanding. More recently, colleagues Ed Doolittle and Lisa Lunney Borden have reenforced the importance of visual, active approaches for Indigenous students. I believe that moving forward together in this way will benefit all students.

I hope this book will extend the excellent existing work on the development of the field of mathematics for teachers from the elementary grades, into the secondary mathematics classroom, and that this contribution will be a first step in that important direction.

INTRODUCTION

MATHEMATICS FOR INTERMEDIATE TEACHERS

Mathematics for teachers

You might wonder why teachers with a post-secondary mathematics background would be interested in learning more about school mathematics. In fact, there has long been the assumption that prospective secondary mathematics teachers already know mathematics. And indeed, they likely do know a great deal about higher level mathematics! However, making the switch from a learner of mathematics, to a teacher of mathematics is a fundamental one. Importantly, one's main focus becomes human development, rather than the mastery of a particular scientific field, albeit a fascinating one. When I made this switch myself, and began thinking about unpacking intermediate level mathematics in such a way that might be useful for students to explore, understand, and generalise the ideas, I realised, with some shock, that I understood mathematics in a mainly functional manner. I could factor expressions, solve equations, and simplify a polynomial. I could work with integer expressions, and multiply and divide fractions. But to my astonishment, I realised that I didn't understand many of the underlying processes or associated reasoning in as deep a way as needed to help students explore and make sense of these ideas.

In parallel with my own personal development, work on what has come to be known as mathematics for teaching (Ball, Thames and Phelps 2008, 389-407) was evolving. One very important aspect of mathematics for teaching is mathematical representation and the associated reasoning, an area in which teachers may need support (Mitchell, Charalambous and Hill 2014, 54-56). Representations can be concrete models such as a model of a fraction process using fraction bars, or models of integer chips, algebra tiles, and so on, but they can also be drawings, computer environments, or even mental images. A crucial aspect of the use of such materials and models is the associated reasoning; that is, an understanding of what the models are for, and how students might use them. Critically, these representations and models are tools to think with, they are not simply a means to show a diagram of a final answer after the thinking has taken place. It is the support of the in-the-moment problem-solving and thinking process that is important.

Much of the work to date on mathematics for teaching has focused on elementary (grades one to six) mathematics. At this level, it has now become fairly acceptable for teachers to use diagrams and physical tools such as counters and other math manipulatives to support students' developing understanding. But by grades seven and eight, and secondary school grades nine and ten, students in the past have typically been expected to work more symbolically. Some teachers even felt that physical tools were only a crutch for less able students (Holm and Kajander 2015, 266-268).

This book extends the understanding of the field of mathematics for teaching to the intermediate secondary level. It is a book on mathematics, not pedagogy. It does not focus on lesson design, questioning techniques, assessment, and so on. But what it may do, is allow and support you to think more deeply about the mathematical ideas in such a way that these pedagogical aspects become much more effective.

While teachers experienced with traditional teaching methods might feel that teaching for understanding, such as by using the tools and methods presented in this book, simply takes too much classroom time, there is great news. Have you ever watched – or even yourself been – a teacher trying, over and over again, to "tell" students a concept that they didn't seem to remember, or didn't want to pay attention to? Repetitive review with unengaged students takes a lot of time, frustrates everyone, and often the concepts still do not make it into students' long term memory. This is because procedures and rules that aren't connected to meaning and reasoning become exercises in memorization, and are easily forgotten (if they were ever absorbed in the first place). In fact, disengaged students often don't take in much useful content anyway. On the other hand, experiences of meaning, ownership, action, involvement, and problem solving can actually motivate students to want to figure things out. Another great aspect of teaching for understanding is that concepts logically build on one another. So new concepts can actually take less time to learn over time, because some of the underpinnings of the ideas are already understood – and retained. I am always astonished at how quickly disengaged students can become engaged if they see a purpose or meaning in a given activity.

In countries such as Canada, attention is also finally being given to the learning needs of Indigenous students. Some scholars (such as Lunney Borden 2018, 64-65) espouse the ideas of visualizing and verbing as helpful starting points. Such learning approaches align well with a models and reasoning approach, as is described in this book. In both cases, students begin with a real world or visual context, concrete materials, and the connection to visual mathematical models. Then the physical materials or models are explored and manipulated according to student reasoning and teacher questioning, to form new ideas. Through discussion, the new ideas can be shared, generalised, and later, formalized and named. Such learning processes also align with the principles of social constructivism, as discussed by Papert, Piaget and others (Papert 1980, vi-viii; Beth and Piaget 1974, 6-23).

If you already like and feel confident in mathematics, you may be surprised when learning more about the mathematical ideas as needed for teaching how much more there is to know about the concepts you thought you already knew! I am always astonished at how frequently even mathematics graduates claim that they are understanding these fundamental concepts as never before. Or, if you are less keen on mathematics, then this book might change your feelings about the subject! Either way, it will certainly help your students.

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CHAPTER ONE

THE PROCESSES OF MATHEMATICAL LEARNING

Mathematics for teachers

This book is intended for teachers of intermediate mathematics. While the main focus of the book is the mathematics needed by teachers of grades seven to 10, the foundations of the relevant ideas originating in the junior grades (grades four to six) are included where needed for the purpose of building on them for the secondary level. As well, what the field calls "horizon knowledge", in other words the idea of where the concepts are heading, is also explored, hence some concepts are connected to the ideas typically developed in grade 11 and 12.

This book is intended for both prospective as well as practicing teachers. In many countries, prospective teachers of high school mathematics have already taken a number of university level mathematics courses. This book does not replace that requirement. Rather, this book focuses on the specific new field of mathematics called *mathematics for teaching*–a type of applied mathematics needed particularly by teachers.

Mathematics for teaching is a specialised field of mathematics, which has been developed over the last 20 to 30 years. Most of the publications in this field only include content up to about grade six, but this book extends the grade levels of the content to the secondary level.

I have experienced a lot of confusion around what the field of *mathematics for teaching* actually is. Some people think it is simply the need for teachers to take more university level mathematics courses. Other people think it is pedagogy – the kinds of topics that would be covered in curriculum and instruction courses. In fact, it isn't really either of these. It is a brand new field in its own right – a particular kind of applied mathematics.

The field of mathematics for teaching was initially described and named by a team of researchers in the northern United States (e.g. Ball et al, 2008). The theory includes two general intertwined aspects, namely subject matter knowledge, and pedagogical content knowledge. The latter of these aspects is the kind of pedagogical work often done in teacher education and preparation. The former, subject matter knowledge, focuses more on aspects of understandings of mathematics, which is the topic of this book. Within subject matter knowledge are common content knowledge, specialised content knowledge, and horizon knowledge. It is likely that you already have good common content knowledge and horizon knowledge; these refer to the knowledge you have already gained from your own previous mathematics courses, related respectively to the curriculum content in the grades in question, and ideas related to where the concepts are heading. However, our research (e.g. Kajander & Holm, 2013; Holm & Kajander, 2020) suggests that even teachers with strong mathematics backgrounds may not have developed specialised content knowledge as needed for teaching. The development of such specialised understanding is the topic of this book.

As an example of the knowledge teachers need, it is likely you already understand that in the number “23”, the symbols “2” and “3” have different meanings due to their *location* within the number. You likely know how to compute 23×14 or $2 – (-3)$. However, perhaps you don’t know how understanding an area model representation of 23×14 allows students to themselves determine ways to simplify expressions such as $(2x + 3)(x + 4)$, without teaching them rules such as “FOIL”. Teachers wishing to build on the area model to develop binomial multiplication need to be aware of the origins of the model with whole numbers.

Importantly, teachers of conceptually-rich mathematics classes need to know about representations, and which models and manipulatives might be helpful for a given context, as well as the associated reasoning. For example, it might be helpful to know which specific fraction division questions can be modelled with fraction strips or bars, such as $\frac{3}{4} \div \frac{2}{8}$, and which questions would be easier with the use of an area model, such as $\frac{4}{5} \div \frac{2}{3}$. You might need to know how integer chips can be used to help students figure out integer operations, and in lesson planning and sequencing, what type of integer questions would require students to know about zero pairs, and which would not. Perhaps you are interested in having students explore how to factor algebraic expressions such as $x^2 + 5x + 6$ for themselves, without imposing a rule, and which representations and manipulatives might be helpful. You might be interested to find out that once students see how the process of binomial multiplication is the “same” as the one for multiplying two-digit whole numbers, they really do not need memorized procedures named with acronyms such as “FOIL” to remind them of the procedure for simplifying algebraic expressions. Even better, the ideas generalise to trinomials without learning new “rules”. If you are interested in these mathematical questions, then this book is for you!

The book does not simply provide a picture, model, or explanation of a known rule, as this would not illustrate a problem-solving approach. Rather, the book *develops* the new mathematical ideas based on familiar ideas, beginning with the concrete and visual concepts and understandings, and then gradually uses those to illustrate how students can abstract and generalise to higher order ideas. Lastly the ideas can be formalized and named. Thus, the "rules" that might otherwise have to be taught directly, become necessary outcomes or generalisations of the models and reasoning approach.

The notion of generalisation in mathematics is a powerful one, deeply linked to the processes of reasoning and proving. This key higher-level concept in mathematics, the idea of using reasoning to move from existing ideas to higher order generalisations, is deeply embedded in the book. Such a process is significantly different from simply using a picture or diagram to "explain" a mathematical procedure, after providing one. It is important to recognise that such "explaining" only takes place after the fact–that is, after "telling" the rule. Rather, the book supports the actual *development* of the method or rule, via models and reasoning. This is a significantly different approach, and much more mathematical, than starting with the rule itself as is often done.

It has been suggested to me many times that this type of mathematical understanding is really *pedagogy*, and thus should be taught in curriculum and instruction courses for teachers. If prospective teachers have not had the opportunity to develop this type of mathematical understanding previously, then these concepts *have* to be included in courses on pedagogy. However, this specialised type of mathematical understanding is indeed *mathematics*, and thus can–and ideally should-be included in stand-alone mathematics for teachers courses, or as the focus of professional development initiatives.

Curriculum reform initiatives around the globe describe the importance of problem-solving, and learning mathematics in a deep, conceptual and contextual way. To effectively teach in this manner requires teachers to have these new and enhanced understandings of mathematics for teaching. Interestingly, teachers' specialised understanding of mathematics as needed for teaching is the one type of teacher mathematical understanding that *has* been linked positively to student achievement (e.g. Baumert et al, 2010). I have also heard over and over the comment "wow, I finally get it now, I never knew all this before!" from prospective teachers when initially experiencing the field of mathematics for teaching.

A models and reasoning approach

Historically in mathematics, the fields of geometry and algebra were closely intertwined. The idea of using a models and modelling approach (Lesh & Doerr, 2003) draws on this historical connection. The word ‘model’ can also be used to refer to a mathematical way of characterizing a real-world phenomenon, in order to try to make sense of it. Climate change, for example, is a context for which mathematical models are used to attempt to predict the future. An alternate use of the word model in mathematical learning is as a visual or concrete representation of a quantity or expression in order to think and reason, and this is the main use of the word model here. In this book, I use the words models and representations somewhat interchangeably, to refer to such visual and often concrete aids to thinking. These aids can take the form of diagrams, physical tools such as math manipulatives, or even mental images. Importantly, these tools will be used as supports to thinking and reasoning – they are much more than diagrams of a final answer as is sometimes thought. As well, their use involves much more than a teacher drawing a picture to “explain” an idea after telling students a rule. Rather, the fundamental idea is to provide a support for students to explore and make sense of an idea themselves, which only later becomes generalised to a method or rule, and only when the students are ready to do so.

Many of the seemingly incomprehensible rules of algebra logically build on the understanding of whole number operations. Ideally, understanding of the reasoning behind these operations should develop in elementary school, and indeed many elementary teachers now do teach much more visually and conceptually. In turn then, it is also necessary for secondary teachers to understand these foundations, in order to explicitly build upon them. In fact, sometimes these foundations need to explicitly be the starting point for the development of flexible algebraic understanding and reasoning. Jo Boaler, one of the most prolific and respected mathematics education researchers of our time, claims that the most important foundation for deep and flexible understanding of algebra is deep and flexible understanding of whole numbers and operations, including methods students come up with themselves (see some of Boaler’s work at www.youcubed.org). Such non-standard student methods are often called “invented methods”. What is important about such invented methods is that in order to come up with the ideas, students need to *reason*.

Reasoning is actually one of the mathematical “learning processes” stated in many Canadian curriculum documents as well as those in other countries, as are representing and problem solving. You may know from your own work in mathematics that reasoning and problem solving often form the mainstays of mathematical thinking and new development. Fundamentally, these processes are what mathematicians *do*. Models and representations are crucially helpful in this regard, in that they provide tools to reason *with*. These tools are far from being supports for students previously perceived as less able, they are fundamental aspects of mathematical understanding and development at all levels of mathematical development.

Many countries have previously imposed what may be termed a traditional Western approach to mathematics teaching, but more recently other ways of thinking and knowing are receiving more attention. In Canada for example, attempts are being made to "Indigenize" learning environments, including mathematics classes. Interestingly, Indigenous ways of thinking and knowing align much better with a problem-based approach which uses representations and reasoning, than with a teacher-directed more procedural approach, even without adding specifically cultural contexts (Lunney-Borden, 2018). This is an exciting benefit of such learning environments.

Mathematics reform

The teaching methods sometimes referred to as "mathematics reform" began to be commonly studied and explored during the 1980's. Research took place in many regions including North America, Great Britain, and Europe. In the United States for example, a document outlining what is involved in this type of teaching was initially published by the National Council of Teachers of Mathematics (NCTM, 1989), and is still often referred to as *The Standards*. The challenge for teachers was that this earlier work was not well-explicated for day-to-day classroom teaching. Of course, teachers wanted their students to deeply understand the ideas, but how exactly was that to be enacted?

Teachers naturally continued to follow the methods they knew and had experienced themselves as learners of mathematics, for want of something else that was well defined. In particular, at the high school level, mathematics teachers have generally themselves been successful in learning mathematics in a traditional teacher-directed, rule-based manner. Such personal bias is difficult to overcome. The best way to open your mind to the possibility of learning mathematics differently is to either experience it yourself, or see a student experience it. Watching a student develop a deep understanding of a concept, and be able to generalise it, is like watching a beautiful sunrise; suddenly all is illuminated! This book is designed to give you the tools to support students' experiences in this manner.

What I find particularly helpful with a focus on representations and models, and the processes of representing and modelling, is that these tools and processes can be used by teachers in both a more directed manner, as well as a more open or problem-based manner. Thus the transition to supporting more problem solving, student inquiry, and alternate and invented methods can evolve gradually, as both teachers and students develop the confidence and problem solving capacity, as supported by experiences in representation and reasoning. You may find yourself learning and developing alongside your students-at least I did!

Inquiry and problem solving

The pedagogical methods often known as problem solving or inquiry are generally drawn from the basic principles of mathematics reform, as based on theories of social constructivism, as just described. It is important to understand that these teaching methods, as the skeptics might suggest, do not imply teachers simply handing students some manipulatives and hoping they will discover something useful. These methods involve carefully planned, crafted, and thought-out lessons, in which student thinking is both anticipated and supported, and possible directions and outcomes are carefully crafted, anticipated, and planned. They require more preparation and involvement on the part of the teacher, along with carefully thought out supports such as visual models and manipulatives, together with an understanding of the associated reasoning. As mentioned, this is not a book on pedagogy; its purpose is not to help you design effective lessons or craft evaluation instruments. Rather, its purpose is to enhance the *mathematical understanding* needed to support this kind of learning environment.

The power of deep mathematical understanding for the purpose of teaching should not be undervalued. You may find that as your own mathematical lens grows ever deeper and wider roots, your values in terms of teaching and learning change in parallel. For example, once the connections among visual representations and meanings of whole number multiplication, binomial expansion, factoring, and other algebraic processes are illuminated, it is hard to think of these things as disconnected algebraic rules with no meaning; rather they become an interconnected web of applications of models of multiplication. Such an evolving view continues to grow, and in turn continues to influence the design and delivery of lessons. The shift to valuing *understanding* is so powerful it has the potential to also shift your classroom practice to one that values flexible thinking, connections, sense making, and deep understanding. And best of all, it has the potential to influence how students see mathematics, as well as themselves mathematically.

The role of coding

Recently there has been a re-emergence of interest in coding as a tool to support mathematical understanding. Indeed, technology provides a dynamic representational tool in its own right. In fact, the ideas around the potential of computer programming, now more commonly referred to as coding, were initially explored by a number of mathematicians and mathematics educators as early as the 1970's. The best known and most prolific of these was Seymour Papert, whose seminal work *Mindstorms: Children, computers, and powerful ideas* (Papert, 1980) ignited the mathematics education community. Papert's work was deeply constructivist in nature, and in a sense far beyond the typical vision of mathematics classrooms of his time. Only now, as the field is getting better at truly enacting the initial promise of constructivism, mathematics reform, problem-based learning, and the use of representations and reasoning including coding, do we see the beginnings of the authentic application of some of these inspirational visions of the future.

Introduction to Scratch coding

One of the issues with early coding environments such as those created by Seymour Papert and his team, was the available technology of the time. Computer storage was no where near as plentiful as now, and as a result computer languages were much less powerful. The children's computer programming language Logo, developed by Papert and his team at the Massachusetts Institute of Technology (MIT) relied on cryptic short forms for commands, such as "FD" for the command to move forward. The commands typically had to be "taught" to students, already disrupting the idea of the environment as a place for children to think and explore. Although quite a bit of research took place in the 1980's around using Logo in mathematics learning, the ideas never really became mainstream.

More recently, the huge technological advances in computer power, combined with research in learning theory resulting in mathematics reform, as well as an emergence of the popularity of technology-based gaming, have all had a role to play in the use of technology environments in mathematics learning. Since the early days of Logo, the MIT team has moved far along with its vision, and more recently their computer language "Scratch", a computer language for children, has become freely available. Early concepts embedded in Logo are now much more user-friendly, dynamic, colourful, and game-like in the new environment, while retaining the original mathematical power. Examples and activities using Scratch will be provided from time to time in the book; the computer language is freely assessable at https://scratch.mit.edu/ and includes many pre-made activities, tutorials, and programs.

CHAPTER TWO

FLEXIBLE UNDERSTANDING OF THE FUNDAMENTAL OPERATIONS: THE FOUNDATION FOR DEEP UNDERSTANDING

Whole numbers and invented methods

It might seem strange to begin a book on intermediate secondary mathematics with an exploration of whole numbers and the four fundamental operations. And yet over and over, it is often the case that difficulties with higher level content are partially rooted in narrow and superficial understandings of the four fundamental operations. In fact, understanding whole number operations in a deep, flexible, and conceptual way turns out to be critically helpful when exploring more challenging concepts such as fraction division, multiplication of algebraic expressions, processes with quadratics, and so on.

In the past, it was thought better to have all students performing arithmetical calculations using exactly the same procedures. They would then practice these over and over, to develop speed and fluency. It turns out that in the age of technology, our priorities shift a bit. At a certain point, when dividing a four digit decimal by a two digit decimal, we will likely reach for a phone or calculator. (And as for the 'what if you are stranded on a desert island' scenario, we will likely have bigger problems than fluency with decimal division). But a number of things remain critically important. First, we need students to have fluent and effective methods for rough estimation – the best way to immediately catch keystroke errors. Second, we need students to understand *why and how* the procedures work, so that the ideas lay the groundwork understanding how to apply the ideas to higher level contexts such as those mentioned. Thus, encouraging multiple methods, as well as students' own invented methods, turns out to be very important.

We often use invented methods when doing mental math, without even realising it. For example, try adding 18 + 25 in your head. After you have finished, jot down your method, and ideally compare with a colleague or classmate. Can you think of other methods? Did you find yourself using the traditional 'regrouping' method, or something else?

It's surprising how many people tend to do this calculation using a method other than the traditional method they were taught, when doing it mentally. For example, some people make the 18 into an easier number by taking 2 from the 25, to add to the 18, making it $18 + 2$ or 20. So the calculation becomes $20 + 23$ instead–much easier. Another common method is to add the bigger parts of each number first, $10 + 20$, and then add on the sum of the smaller parts $(8 + 5)$. And there are many more. All of these invented strategies are both great for mental math and also lay the foundation for a more flexible understanding of operations, in order to build algebraic reasoning in the future.

Activity 2-1:

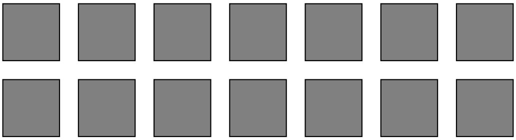
1. Try these calculations with mental math, and afterwards, record your method. If possible, do each in more than one way. Ideally, share your methods with a partner.
   a) $197 + 254$
   b) $306–189$
2. A student calculated $306–189$ by calculating $307–190$. Does this method work in general? Why or why not? Explain the student's possible reasoning.

Models and manipulatives for whole numbers

Concrete models and manipulatives are visual representations and physical tools to help with the reasoning process. This modelling and reasoning process is much more than "showing your work" with a picture, after completing a calculation. Such tools and reasoning processes are aids to thinking. Appropriate representations can be constructed by students themselves, but sometimes teachers need to support the process. Thus, teachers need to have a toolkit on hand, and even more importantly, have an understanding of possible student-generated representations and how these models can support students' thinking.

Counters are one of the most fundamental manipulatives available. Conveniently, we are equipped with ten of them usually, right on our hands! However, as the numbers get larger, it is simply inefficient to count or draw a lot of counters, one by one.

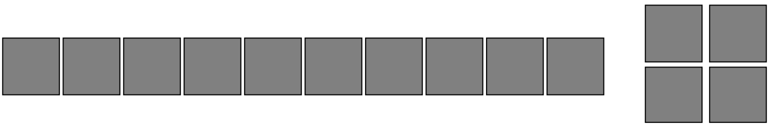
Fig. 2-1:



So it makes sense to *group* the counters, and since we normally have ten fingers, a popular way of grouping involves groups of *ten*; the *decimal* system. Perhaps if we lived on a planet where beings had eight fingers we would have a system based on groups of 8. Such an *octal* system would be much more efficient for computer programmers as computers work on a *binary* number system, or base 2, which is much easier to translate to base 8 then base 10. But let's stick with the modern standard base 10 for now! (However, if you really want a challenge, try practicing some standard calculations using a different base.)

Here is the same quantity shown above, but now grouped by tens.

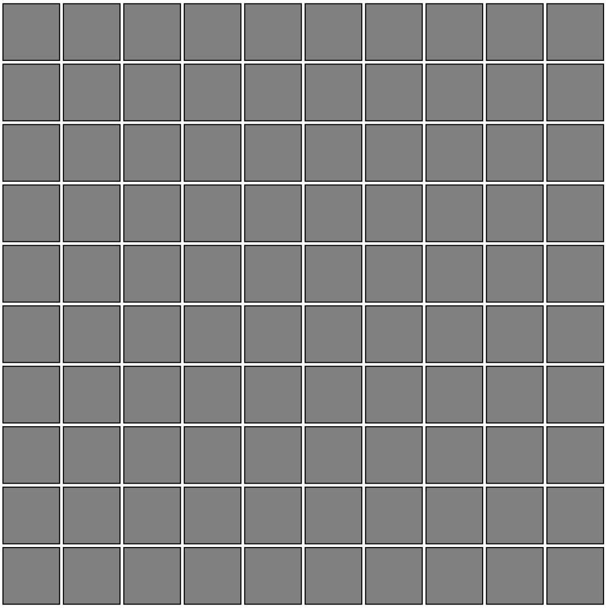
Fig 2-2:



We have one group of 10, and 4 units, so we write 14. The value 14 as shown has four "units", and one "ten" (the tens piece is sometimes nicknamed a "long", or a "rod").

This "place value" system, in which the *location* of the digit matters as well as its size, is typically developed with children in early elementary grades. Teachers may use many representations to support the ideas of place value, but one of the standard types of materials are the "base 10 blocks", illustrated here. To represent even more counters, ten groups of ten (10 x 10), can represent a group of 100, shown by ten of the longs. This "hundreds" piece is comprised of a 10 by 10 arrangement of unit squares.

Fig. 2-3:



Sometimes this 100's piece is nicknamed a "flat". The largest piece we can build in our three-dimensional world is the *thousands cube*, sometimes called the "block". (Imagine 10 flats stacked on top of each other to make a 10 by 10 by 10 cube). Sometimes we write $10 \times 10 \times 10$ as 10^3, which is read as "ten-cubed". Yes, the meaning of the word *cubed* is literal!

These base 10 materials are not only a fundamental representation of our number system and its properties and operations, but they also *directly* generalise to representations of algebraic representations. In fact, the algebra-tile representations used in later chapters specifically build on a meaningful understanding of whole number representations and operations. If students don't have a solid and conceptual foundation of the whole number system and of operations on it, subsequent algebraic operations and "rules" will have little meaning – just as some people report experiencing when learning high school algebra. I often found myself reviewing these whole number expressions and operations when teaching grade nine mathematics, prior to building algebraic understanding.

Additive operations: Models to numeric methods

If you watch primary age children exploring basic operations such as addition and subtraction without being taught formal "rules", you will be astonished how many great methods they can come up with. (See for example, Constance Kamii's book *Children Reinvent Arithmetic*). These ways not only increase overall number sense, they are also very useful for mental math facility, as the $18 + 25$ example illustrated.

A models and reasoning approach can also be used to develop the "traditional" or standard addition and subtraction procedures. It should be noted however that unlike some classroom perceptions, these "standard" procedures are somewhat culturally-based. For example, I have taught students from a number of other countries who were taught a different "standard" subtraction procedure than the usual North American one.

Let's try the $18 + 25$ calculation again with base ten blocks. Ideally, follow along with concrete materials, drawings, or with a virtual manipulatives platform.

The fundamental action associated with addition is *combining*. So to model the addition operation, we can model each of the amounts 18 and 25 with base ten blocks and then combine the amounts into one representation. Already we see that the order of the traditional method of starting with the units or ones is not likely the method we would choose if we were doing mental math, such as when shopping. If we were adding 18 and 25, we would most likely add the 10 and 20 first, which are the most significant parts of the numbers, or possibly first adding 2 from the 25 to the 18 to make 20 and 23. (If you're like me, you might find that by the end of this chapter, your reliance on these standard methods might have shifted to ones you find more useful!).

See if you can model the standard method for the calculation 18 + 25 using base ten blocks (or drawings of them). Pay particular attention to the decomposing step, in which a group of ten units is "traded" for a "long". While the term "regrouping" was used in the past, teachers are now being encouraged to use the more mathematically accurate term "recomposing" for this same process.

Once you have created the model, see if you can make an explicit link to the traditional numeric procedure. You may now find the writing of the recomposed amount (often a small "1" at the top of a column) is something that could easily cause a mistake.

Ex. 2-1

$$\begin{array}{r} 1^{1}8 \\ +2\ 5 \\ \hline 4\ 3 \end{array}$$

Based on the model, I have found myself writing traditional addition calculations like this instead:

Ex. 2-2

$$\begin{array}{r} 18 \\ +25 \\ \hline 13 \\ 30 \\ \hline 43 \end{array}$$

13 ←the "recomposed" ten, written as the actual value rather than a tiny "1" above

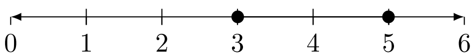
Activity 2-2:

Try adding 136 + 278 in as many ways as you can. You might use an invented numeric method or two, a model, and the standard method.

While addition involves *combining*, subtraction can represent a number of actions. Brainstorm a quick list of what subtraction might mean.

You might have listed interpretations such as *remove* (or *take away*), *compare* or *how far apart*. It is important that students have seen all these ideas in early years, because all will be useful later. For example, while *take away* is often an early way to think about subtraction, ensuring that students see other problems such as those involving change, comparison or distance will become very important later, such as with integer subtraction. For example, the 5 – 3 operation can be thought of as "how much farther is 5 than 3" on the number line:

Fig.2-4



The example in Activity 2-1, #2 illustrated a student calculating 306-189 by calculating the easier answer to 307–190. This method can be understood if we simply think of sliding the two numbers one unit to the right on the number line–the distance between them is preserved. A student using this method and asked to "show their work" might well draw a number line with the values indicated, which is an excellent way to illustrate this thinking.

It is critical for teachers to understand that just because we have done a particular traditional procedure many times and thus find it "easy", students won't necessarily find the traditional method the easiest one. In fact, as with the number line example just discussed, sometimes it isn't! For example, the calculation 1001–999 would be very difficult with the standard procedure– and yet is simple when we think about comparing, or adding up from 999. In fact there are other "standard" methods than the typical North American ones. I have met students from other countries who have learned a different "standard" method for subtraction. Try it with $123 - 86$:

Ex. 2-3

$$\begin{array}{r} 1^{1}\,2^{1}\,3 \\ -\,{}^{1}\;\not{8}^{9}\,6 \\ \hline 3\;\;7 \end{array}$$

Can you figure out what is going on? If not, you may be feeling like many students first feel when first presented with the standard method in North America. We'll come back to this method after exploring other ways to subtract.

Activity 2-3: Further explore 123–86 using

1. An invented numeric method of your own
2. A model with base ten blocks
3. The standard North American numeric procedure
4. The equal additions method above. (Hint – the first step which makes "3" into "13" is compensated for by *subtracting* an extra ten later – the 80 to be subtracted becomes 90.)

As with addition, students are very good at coming up with their own methods, and these methods can be helpful and generative for developing later concepts. For example, a grade 6 teacher recently shared this method with me which came from one of her students:

Ex. 2-4

$$123–86 = 40–3 = 37$$

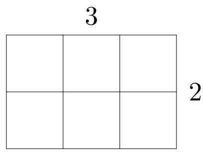
Can you figure out the student's thinking? (See Hint 1 at the end of the chapter if you are stuck!)

Multiplicative operations: The area model

One of the ways that multiplication is often introduced is by "repeated addition". While it is true that 2×3 can be thought of as "two groups of 3" or "2 added three times" or even "three groups of 2", these ways tend to rely at least partly on additive reasoning. However, if we are thinking of "two groups of 3", we don't model both the 2 and the 3 as separate values with counters as when adding. Rather, we start with 3, and then the "$2\times$" operates on the 3. This thinking will be important when thinking about multiplying fractions. In fact, there is more to multiplicative reasoning than adding groups. Take a moment to brainstorm all the ways you might think about multiplication.

Multiplicative reasoning relates to area, dimensions, ratios, proportion, division, and many other things. It is fundamental to a conceptual understanding of algebra. Here is a very important model called the "area model" representation of 2×3.

Ex. 2-5



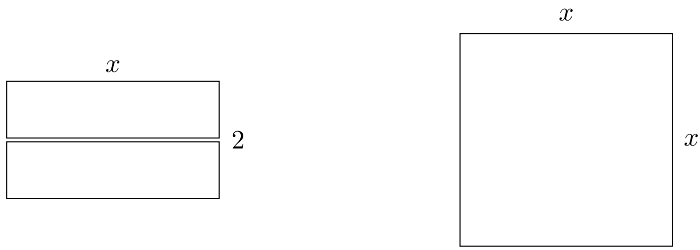
The area model is a critically important model to illustrate the product of two quantities, and continues to be useful in higher grades. The area model has an immediate connection to rectangular area, hence its name. The two *factors* (here, 2 and 3) form the measures of the sides of the rectangle, and the associated rectangular area represents the *product*.

The area model also makes it clear that multiplication can be done in either order. We see that multiplying by something can be thought of as increasing the dimension of the original quantity. This illustrates "multiplicative thinking". In the area model, each value forms the side length of a rectangle, and the answer is the area inside the rectangle formed.

Multiplicative reasoning is also critically important for understanding fractions. For example, if a student seeking to model $\frac{1}{2} \times \frac{2}{3}$ begins by modelling *both* the quantities, here one-half and two-thirds, they may be thinking additively. Alternately, if they are able to think "one half *of* two-thirds", then they may begin by representing only the two-thirds quantity (perhaps with fraction manipulatives), and then modelling the multiplication by taking *one-half* of that amount. This sequence involves *multiplicative* thinking, which should be conceptually developed with whole numbers prior to working with other numbers and quantities.

When working with whole numbers, the area model can in fact be used to *generate* the traditional multiplication procedure. But even more importantly for intermediate teachers, it can also provide a basis for the construction of fraction multiplication, multiplication of algebraic expressions, and later, even algebraic techniques such as factoring, simplifying and so on. Visual models such as this can also be helpful in helping students understand expressions such as $2x$, and also x multiplied by x, (sometimes written x^2). Grade 9 teachers working with students who have been taught traditionally often notice students confusing x^2 with $2x$. The critical difference here is that the first term, x^2, involves *multiplication* by x while the second, $2x$, can be thought of as *adding* x's, or multiplying by 2.

Fig. 2-5

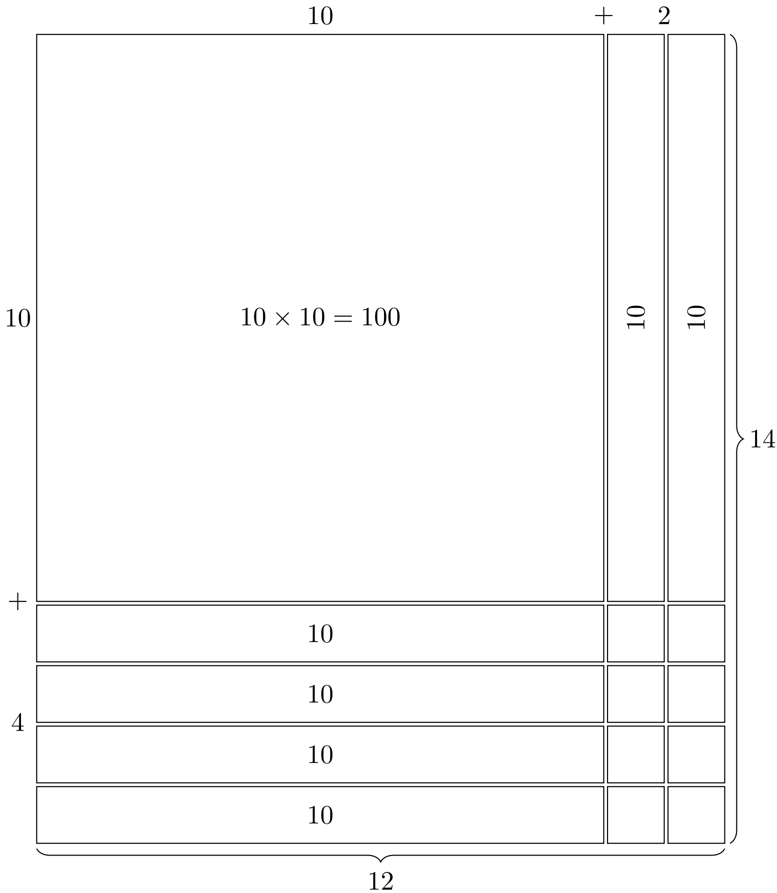


Activity 2-4:

Use an invented numeric method to answer 12×14. Now redo the calculation with the standard numeric method you remember being taught in school. Thirdly, use drawings, or base ten blocks, to model 12×14 with an *area* model. Remember, the 12 and 14 form the length of the sides of the rectangle, when forming an area model.

Let's further explore the *area* model, and explore how that model could actually be used to generate the traditional procedure with students (rather than "explaining" to students the "steps" in the procedure, after presenting the procedure to students).

Perhaps your area model in Activity 2-4 looked something like this:
Fig. 2-6



The calculation 12 × 14 can be thought of as four separate calculations, as can be found in the area model above. The following somewhat simplified version of the area model shows how the four sub-products are generated. These can be added up in any order.