Fourier Analysis and Medical Image Filtering

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By

M'hamed Bentourkia

Cambridge
Scholars
Publishing

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This book first published 2022

Cambridge Scholars Publishing

Lady Stephenson Library, Newcastle upon Tyne, NE6 2PA, UK

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

ISBN (10): 1-5275-8926-9
ISBN (13): 978-1-5275-8926-1

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About the Author

M'hamed Bentourkia, Ph.D., is a nuclear physicist with undergraduate and graduate studies completed at University of Quebec at Chicoutimi (B.Sc.), University of Montreal (M.Sc.), and University of Sherbrooke (Ph.D.), Quebec, Canada. He worked as a postdoctoral fellow at Catholic University of Louvain-La-Neuve, Belgium and at Hammersmith Hospital, London, England. He is a professor at the University of Sherbrooke since 2000. His main research domain are positron emission tomography image analyses and pharmacokinetic modeling, and Monte Carlo simulation in medical imaging and in radiation dosimetry. He formed several M.Sc. and Ph.D. students as well as undergraduate trainees (B.Sc.). He published more than 100 publications and several conference abstracts and book chapters.

PREFACE

This book presents the theory necessary for the understanding, implementation and use of Fourier series and Fourier transforms. The content is aimed at both beginners and people familiar with the theory of Fourier analysis. The theory is presented in the simplest and most focused manner possible so as not to weigh down the content with generalities that span multiple disciplines. We have avoided burdening this content with rigorous mathematics such as definitions and names of sets, theorems and other postulates. We have supported the theory with several examples, solved exercises and programming in Matlab throughout the content. We have privileged the understanding of Fourier transforms which is done by calculations and applications rather than by analytical developments.

The programs written in Matlab in a simple and concise way allow even those who have never used Matlab to apply them. These programs are accompanied by an introduction to Matlab. The main purpose of these programs, other than to display the results, is that the user can implement them by modifying certain parameters in order to observe their effects, and this allows the reader to better understand the influence of these parameters. In other words, the reader can use these programs for simulation purposes, which will increase their understanding of both the mathematics and the phenomena intended by the applications.

Usually equations are written as two members per line or they are separated by punctuation. In this book, we have written the equations repeatedly where multiple members are separated by an equal sign (=) and may span more than one line in some cases. This choice is made so as not to repeat the handwriting of the members with each mathematical development and also to save space. From a reading point of view, we believe that it is easier to move from one member to the next when the equations are written in chain. Additionally, we have written as many reasonable steps as possible in solving equations to allow for easy switching from one equation to another.

The examples and exercises provided in this book are among the simplest in order to understand the theory and facilitate their programming. Some exercises are classical and elementary and can be found in several references. The reference books for this work that we have consulted most often are those of Morrison (Introduction to Fourier analysis. Norman Morrison, 1994) [1], of Hsu (Applied Fourier analysis. Hwei P. Hsu, 1984) [2], of Howell (Principles of Fourier analysis. Kenneth B. Howell, 2001) [3], of Spiegel (Analyse de Fourier et application aux problèmes de valeurs aux limites. Spiegel, Murray R, 1974) [4] and of Bracewell (The Fourier transform and its applications. Ronald Bracewell, 2000) [5].

For safe and effective practices, it should be noticed that the texts, equations and applications in this book do not commit the author or the publisher in any way as to their accuracy and that users should verify this before any use in particular applications.

SUMMARY

Fourier series and Fourier transforms are widely used in several fields for signal and image processing. Stated in a simple way, Fourier series allow to represent functions as a sum of cosine and sine functions. This representation is very useful especially in the case of numerical functions. On the other hand, Fourier transforms allow to discriminate functions according to their frequency. In this handbook, the understanding of Fourier series begins with the introduction of complex numbers in their different forms: algebraic, trigonometric and exponential. The analytical, discrete and fast Fourier transforms are initially described by simple numerical procedures in order to follow the different operations and to be able to visualize the correspondence between time or space and the frequency domain. The origin of filters is demonstrated with electronic circuits. Filters determined with Fourier analysis and other filters related to the frequency domain are studied and applied to functions and medical images of different formats. All the concepts are presented with basic analytical and numerical examples, backed with exercises with direct calculations and with programming in Matlab, and by depicting in 185 figures of diagrams, graphs and images. The simplicity in the introduction of the definitions helps the reader to conceptualize the theory and to understand its application. This handbook is intended to be a guide for students, teachers and researchers.

Chapter I. Complex Numbers

I.1 Brief history of complex numbers

Imaginary numbers first appeared in the 1st century AD when the Greek mathematician Heron of Alexandria was trying to calculate the volume of a truncated pyramid and had to calculate the number $\sqrt{81-144}$. He had then evaded the problem by eliminating the negative sign under the radical.

It was not until the 16th century that imaginary numbers reappeared when Italian mathematicians tried to solve equations of the third degree and above. Scipione del Ferro (1465 - 1526) considered the solutions to equations with square roots of negative numbers as impossible [6]. Nicolo Fontana, called Tartaglia (1500 - 1557), Girolamo Cardano (1501 - 1576), Ludovico Ferrari (1522 - 1565) and Raphaël Bombelli (1526–1573) [7] faced the same dilemma and recognized the existence of square roots of negative numbers, but they did not come up with solutions. They solved the equations of the third degree according to the method attributed to Cardano: $x^3 + px + q = 0$, and that of the 4th degree according to the Ferrari method: $ax^4 + bx^3 + cx^2 + dx + e = 0$. They found solutions to these equations having their square as a negative number. Later, in 1637, René Descartes named these solutions imaginary solutions. In the 18th century, Abraham de Moivre and Leonhard Euler consolidated the concept of complex numbers [6].

I.2 Definition

The problem posed to the 16th century algebraists was to solve second and third degree equations.

Let the equation of the second degree consists of the real constants a, b and c, and the variable x (eq. I.1):

$$ax^2 + bx + c = 0 \qquad \textit{eq. I.1}$$

It is to calculate the values of the unknown variable x which allow to verify this equality. Since the variable is of degree 2, which is the highest exponent of the variable x, then this equation admits two solutions, x_1 and x_2 so that these two values verify eq. I.1:

$ax_1^2 + bx_1 + c = 0$ and $ax_2^2 + bx_2 + c = 0$.

To calculate the solutions (roots) of this equation, we define the discriminant D:

$$D = b^2 - 4ac \qquad \textit{eq. I.2}$$

The roots are then given by:

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \textit{eq. I.3}$$

According to this notation, the root x_1 is obtained with the $+$ sign preceding the square root in the numerator, and x_2 is obtained with the $-$ sign. There is no inconvenient in assigning to x_1 the $-$ sign and to x_2 the $+$ sign. The essential is to find the two roots x_1 and x_2.

In eq. I.3, the quantity under the square root can be negative, which drives us to the imaginary numbers, because it is not possible to have a negative value as a square of a real number. If the discriminant D is negative, we can rewrite it as $D = -1 \times D1$, with $D1 > 0$. Thus, it is possible for us to calculate the square root of the positive number $D1$, but we need to calculate the square root of -1 in order to be able to calculate the square root of the discriminant D. It is at this level that the imaginary numbers intervene, by posing:

$$i^2 = -1 \qquad \textit{eq. I.4}$$

Thus, the discriminant D can be expressed in the form:

$$D = -D_1 = -1 \times D_1 = i^2 \times D_1$$

Now, with this new definition, it is possible to calculate the square root of the discriminant D, whether positive or negative. If $D > 0$, its square root is real, and if $D < 0$, its square root is imaginary, i.e. it contains the imaginary number i.

As long as we can transform -1 into i^2, we are able to calculate all the roots of polynomials of different orders.

For example, the square root of a real number a: $\sqrt{a^2} = \pm a$. Example: $\sqrt{2^2} = \pm 2$. If the number is negative: $\sqrt{-2^2} = \sqrt{i^2 2^2} = \pm 2i$.

After we have introduced the imaginary numbers, let us go back to solving eq. I.1. The solutions or roots x_1 and x_2 are indicated by eq. I.3. We can predict three types of roots according to the sign of the discriminant D:

1. $x_{1,2}$ are real and distinct roots if $D > 0$.
2. x_1=x_2 are real roots if $D = 0$.
3. $x_{1,2}$ are complex conjugate roots if $D < 0$. Here the conjugate word means the imaginary parts have opposite signs, these are the signs $\pm$ which are in front of the square root of the discriminant in eq. I.3. A number that contains an imaginary part is called a complex number. We will come back to this a little later.

Example I.2.1

Find the roots of $x^2 - 3x - 4 = 0$. Answers. This polynomial is of the form $ax^2 + bx + c = 0$ where $a = 1,\ b = -3,\ c = -4$.

The discriminant value is: $D = b^2 - 4ac = (-3)^2 - 4 \times 1 \times (-4) = 25$.

The roots are given by eq. I.3: $x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-3) \pm \sqrt{25}}{2}$, as $x_1 = 4$ and $x_2 = -1$. Let us see if these two roots satisfy the given equation. Let us use first x_1: $x_1^2 - 3x_1 - 4 = 0$. Replacing x_1 by its value: $4^2 - 3 \times 4 - 4 = 0$, which is true, the equation value is 0. Now let us replace x par x_2: $x_2^2 - 3x_2 - 4 = 0$, giving $(-1)^2 - 3 \times (-1) - 4 = 0$ which evidently gives 0.

Example I.2.2

Find the roots of $x^2 - 3x + 3 = 0$. Answers. This polynomial is of the form $ax^2 + bx + c = 0$.

The discriminant is: $D = b^2 - 4ac = (-3)^2 - 4 \times 1 \times 3 = -3$.

We find that $D < 0$, this means that the square root of the discriminant gives an imaginary number, and the solutions are two complex conjugate numbers given by $x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-3) \pm \sqrt{-3}}{2}$ which are $x_1 = \frac{3}{2} + i\frac{\sqrt{3}}{2}$; $x_2 = \frac{3}{2} - i\frac{\sqrt{3}}{2}$.

The form of these two roots informs us about two notions:

1. these two roots appear identical except that the signs in front of the imaginary part are opposite, a $+$ for x_1 and a $-$ for x_2. This is what we earlier called complex conjugate roots.
2. The two roots are made up of two terms. The first is real, that is the number $\frac{3}{2}$, and the second is imaginary because it is multiplied

by i, that is $\pm i\frac{\sqrt{3}}{2}$. The sum of these two terms forms a complex number. A complex number is therefore formed of a real part and an imaginary part, that is in a general form: $z = a + ib$, with a and b belonging to the set of real numbers $\Re$ $(a, b \epsilon \Re)$.

Application with Matlab

Since the calculations are quick and easy in the examples Example I.2.1 and Example I.2.2 and they do not require writing a program (function), it is faster to do the calculations in the Matlab command window (Matlab command window, which we abbreviate as MCW). It suffices then to type the following texts, according to the examples:

```
p=[1 -3 -4], r=roots(p),
```

and Matlab displays:

```
p =  1    -3    -4
r =  4     -1
```

Here, we have defined a vector p in which we put the values 1, -3 and -4 which are the coefficients of our polynomial in Example I.2.1. Next, we used the Matlab function roots which allows to calculate the roots that we called r. The Matlab display gives the values of p and r. If we followed the equation of p and r with a semicolon instead of a comma, Matlab would not have displayed anything: p = [1 -3 -4]; r = roots (p); and in this case, by typing the name of the variable p or r, Matlab displays its value.

The way you do the calculations in MCW does not allow you to remember these calculations or to reuse them on another day. It is also not convenient to type long text. We then proceed by creating a text file by typing in MCW: edit example_I1. Matlab then displays a window to confirm the creation of a new file. We click on YES to create a new file. The new file opens in the Matlab editor. We type the following text:

```
function [p,r]=example_I1
p=[1 -3 -4];
r=roots(p);
```

Then, we save the file by clicking on $File\ -\ Save\ as$ and choosing the name example_I1.m and the directory C:\user\fourier where to save the file. We see that the edited file has an .m extension associated with it.

To be able to run the example_I1.m file, we must first tell Matlab in which folder this file is located with the addpath command, by typing in MCW: addpath C:\user\fourier -end if for example your file has been saved in the C:\user\fourier directory. The declaration with addpath is only done once during a work session so that Matlab can locate the file.

To run the newly created file, we type in MCW: [p, r] = example_I1; then, to display the results, we type the variables p or r.

To implement Example I.2.2, we take the same procedure as for Example I.2.1: edit example_I2;. Once the example_I2.m file has been created, enter the following text in it then save the file:

```
function [p,r]=example_I2
p=[1 -3 3];
r=roots(p);
```

To run the file, in MCW, type: [p,r]=example_I2, and Matlab displays:

```
p =
    1    -3     3
r =
   1.5000 + 0.8660i
```

1.5000 - 0.8660i

We now want to generalize the program example_I1.m and call it example_roots1.m to do the same calculations as example_I1.m and example_I2.m. First, we create the program example_roots1.m by opening example_I1.m, then saving it with this new name, example_roots1.m. In example_roots1.m which is already open, remove the declaration for p, and keep only the line r=roots (p);. We must also replace the header of the function function [p,r]=example_I1 with function r=example_roots1(p). So we provide the variable p to example_roots1.m and the program returns the variable r. If the file is not open, we first make sure that the folder or directory where this file is located is in the list of directories of Matlab. To see this list, type the path command in MCW. If it isn't, add it by typing in MCW: addpath C:\user\fourier -end, where C:\user\fourier is the name of the directory where the program is located. If it has been closed, the file can be opened by typing in MCW the name of the file: edit example_roots1. To run the example_roots1.m program, we need to supply it with the vector p. This is done as follows in MCW: p = [1 -3 -4]; r = example_roots1(p);. To see the result, we type in MCW r. The example_roots1.m program or function can now calculate any roots of polynomials by entering the values of p.

Possible confusions

In some works, j represents the complex value instead of i, as in electricity, where the letter i designates the intensity of the electric current.

The letters i and j could be used as the unit vectors of a coordinate system in 2D, in this case, we choose u and v as unit vectors (or e_1 and e_2 etc.).

In some software, the letter i is reserved for the complex unit value (as in Matlab), we must distinguish it from the variable used in the calculation loops.

Also, Matlab recognizes j as much as i as a complex number. By typing in the command window i, or j, Matlab displays:

```
i
ans =        0 + 1.0000i
j
ans =        0 + 1.0000i
```

By definition, a complex number is a number composed of a real part and an imaginary part: $z = a + ib$ where a is the real part and ib is the imaginary part because it contains the imaginary i. To start, we will consider the coefficients a and b as real constants.

I.3 Geometric representation of complex numbers

The complex number $z = a + ib$, with a and b real, can be written in a compact form $z = (a, b)$. This writing reminds us of writing a point in a coordinate system according to its coordinates along the x and y axes. Also, we can call a, the real part of z, $a = Re(z)$, and b, the imaginary part of z, $b = Im\,(z)$ (with Matlab: z=4+6i; real(z) displays 4 and imag(z) displays 6).

By plotting the complex number z in a frame formed by the axis of real numbers represented by the usual x-axis, and by the imaginary number axis represented by the usual y-axis. Thus, the coordinates of z are a, along the real axis, and b, along the imaginary axis. Fig. I-1 presents the complex number $z = 2 + 3i$, where $a = 2$ and $b = 3$.

A complex number is characterized by its coordinates $a = 2$ and $b = 3$ in Fig. I-1, by its modulus $|z|$ which is given by $|z| = \sqrt{a^2 + b^2} = \sqrt{2^2 + 3^2} = \sqrt{13}$ according to the Pythagorean rule, and finally by its argument, that is, by its angle made by the vector starting from the coordinates $(0{,}0)$ towards the coordinates $(2{,}3)$ with the axis of the reals, as $\theta = atan\left(\frac{b}{a}\right) = atan\left(\frac{3}{2}\right)$, where $atan$ is the arc tangent.

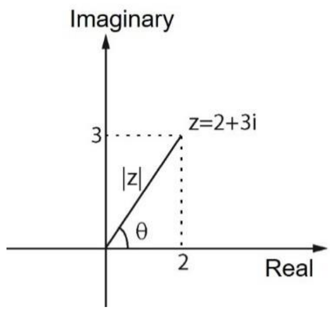

Fig. I-1. Graphic representation of a complex number. The real part is carried by the x-axis, and the imaginary part is carried by the y-axis.

Note that we write $\theta = atan\left(\frac{b}{a}\right)$ and the result is given in degrees most of the time (in Matlab we have to write θ=atand (b/a) to get the angle in degrees, and θ=atan(b/a) gives the angle in radians. Example: d=atand(3/2) gives 56.31 degrees, r=atan(3/2) gives 0.98 radians and d=r*180/pi gives 56.31 degrees). In the following, we write $atan$ for both degrees and radians. In other cases where the expression in radians is usual, we use the angles in radians. A complex number can therefore be represented by its coordinates a and b, or by its modulus $|z|$ and its argument θ. The relation between these two representations is given by:

$$|z| = \sqrt{a^2 + b^2}$$
$$\theta = atan\left(\frac{b}{a}\right) \qquad \textit{eq. I.5}$$

Note that it is possible to calculate the argument using the modulus and the sine and cosine functions:

$$|z| = \sqrt{a^2 + b^2}$$
$$\theta = \text{acos}\left(\frac{a}{|z|}\right)$$
$$\theta = \text{asin}\left(\frac{b}{|z|}\right)$$

If you use the latter option, be sure to calculate the argument with the cosine and sine to determine the sign of the angle.

Example I.3.1

Let the complex number be $z = 3 + 5i$. a) Calculate its modulus and its argument. b) Represent z graphically. Answers. a) The modulus of z is: $|z| = \sqrt{3^2 + 5^2} = \sqrt{34} = 5.8$.

The argument of z is: $\theta = atan\left(\frac{5}{3}\right) = 59°$.

b) Graphic representation of z.

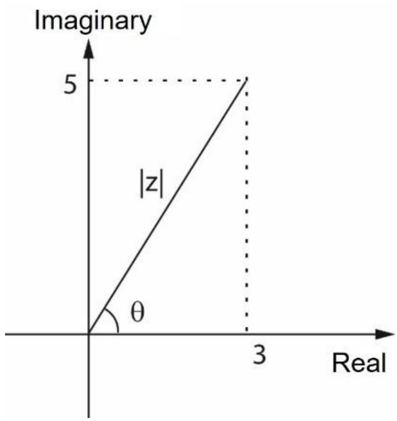

Fig. I-2. Graphical representation of $z = 3 + 5i$. The real part is carried by the x-axis and is 3, and the imaginary part is carried by the y-axis and is 5.

By convention, positive angles are counted in the trigonometric direction, i.e. counterclockwise. Negative angles are counted clockwise. The origin of the angles being the x-axis or the real axis in the case of complex numbers. Fig. I-3 shows two angles θ and ϕ, with $\theta > 0$ and $\phi < 0$. These two examples are illustrated by the complex numbers $z_1 = 3 + 5i$ and $z_2 = 3 - 5i$.

The arguments of z_1 and z_2 are calculated as before with the formula $\theta = atan\left(\frac{5}{3}\right) = 59°$ and $\phi = atan\left(\frac{-5}{3}\right) = -59°$. On the other hand, the moduli of z_1 and z_2 are the same: $|z| = \sqrt{3^2 + 5^2} = \sqrt{3^2 + (-5)^2} = \sqrt{34} = 5.8$.

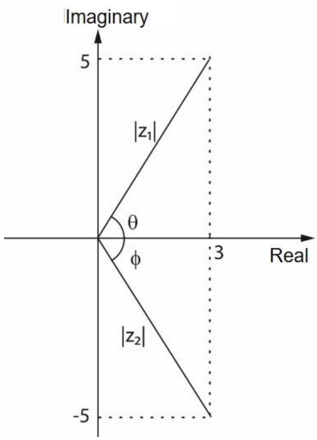

Fig. I-3. Graphical representation of $z_1 = 3 + 5i$ and $z_2 = 3 - 5i$ with their moduli $|z_1|$ and $|z_2|$, as well as their angles θ and ϕ.

We note that these formulas do not tell us about the true position of z. Let us compare the moduli and arguments of the numbers $z_1 = 3 + 5i$ and $z_4 = -3 - 5i$. These two numbers have the same moduli and arguments calculated by the formulas $|z| = \sqrt{a^2 + b^2}$ and $\theta = atan\left(\frac{b}{a}\right)$, either $|z_1| = |z_4| = 5.8$ and $\theta_1 = \theta_4 = 59^0$, and they would be superimposed in the graph. The same thing happens to the numbers $z_3 = -3 + 5i$ and $z_2 = 3 - 5i$, $|z_3| = |z_2| = 5.8$ and $\theta_3 = \theta_2 = -59^0$. On the other hand, the coordinates clearly indicate different positions of these four numbers z_1, z_2, z_3 and z_4 on the chart (Fig. I-4).

Since the argument as calculated by $\theta = atan\left(\frac{b}{a}\right)$ indicates only the angle that the complex number makes with the axis of the reals, we must refer to the geometric representation to calculate the argument. Fig. I-4 shows the arguments of the following complex numbers as calculated by $\theta = atan\left(\frac{b}{a}\right)$: $z_1 = 3 + 5i$, $z_2 = 3 - 5i$, $z_3 = -3 + 5i$, and $z_4 = -3 - 5i$, and Fig. I-5 shows the true arguments of these same numbers based on the graphs. The arguments of Fig. I-5 are: $\theta_1 = 59^0$, $\theta_2 = 360^0 - 59^0$, $\theta_3 = 180^0 - 59^0$, $\theta_4 = 180^0 + 59^0$.

These same angles can be expressed in another way using positive and negative angles at the same time (here we are talking about angles and not arguments): $\theta_1 = 59^0$, $\theta_2 = -59^0$, $\theta_3 = 180^0 - 59^0$, $\theta_4 = 180^0 + 59^0$. What emerges from these new expressions are the symmetries with respect to the axis of the reals. Thus z_1 and z_2 are symmetrical, and the same for the pair z_3 and z_4. This observation is also present in the algebraic expressions of these four numbers where the imaginary parts are of opposite signs, so z_1 and z_2 are complex conjugate numbers, and likewise for z_3 and z_4.

Clearly, the four numbers have identical moduli, either $|z| = \sqrt{3^2 + 5^2} = \sqrt{3^2 + (-5)^2} = \sqrt{(-3)^2 + 5^2} = \sqrt{(-3)^2 + (-5)^2} = \sqrt{34} = 5.8$, and separate arguments that distinguish them as set out in Fig. I-5: $\theta_1 = 59^0$, $\theta_2 = 360^0 - 59^0 = 301^0$, $\theta_3 = 180^0 - 59^0 = 121^0$, $\theta_4 = 180^0 + 59^0 = 239^0$.

We can establish a simple rule to calculate the argument of a complex number $z = a + ib$. Let be the angle $\alpha = atan\left(\frac{b}{a}\right)$, the argument of z is:

$$\theta = \alpha, \qquad if\ a \geq 0, b \geq 0, \quad 1^{st}\ quadrant$$
$$\theta = 180 - \alpha, \qquad if\ a < 0, b \geq 0, \quad 2^{\text{nd}}\ quadrant$$
$$\theta = 180 + \alpha, \qquad if\ a \leq 0, b < 0, \quad 3^{\text{rd}}\ quadrant$$
$$\theta = 360 - \alpha, \qquad if\ a > 0, b < 0, \quad 4^{\text{th}}\ quadrant$$

Obviously, a complex number whose imaginary part is 0 is a pure real number, and a complex number whose real part is 0 is a pure imaginary number. For example $z_1 = a$ and $z_2 = ib$. The graphical representation of a purely real number has only one component along the axis of the reals, and the purely imaginary number has a component along the axis of the imaginaries. The modulus is the component itself, i.e. $|z_1| = \sqrt{a^2} = a$ for $z_1 = a$, and $|z_2| = \sqrt{b^2} = b$ for $z_2 = ib$, while the argument $\theta = atan\left(\frac{b}{a}\right)$ could encounter specific difficulties. Indeed, for $z_1 = a$, $\theta = atan\left(\frac{b}{a}\right) = atan\left(\frac{0}{a}\right) = 0$. This can be verified graphically where the angle is along the axis of the reals. For $z_2 = ib$, the argument calculated according to the formula $\theta = atan\left(\frac{b}{a}\right) = atan\left(\frac{b}{0}\right) = 90^0$. Note that the expression $\frac{b}{0}$ is undetermined.

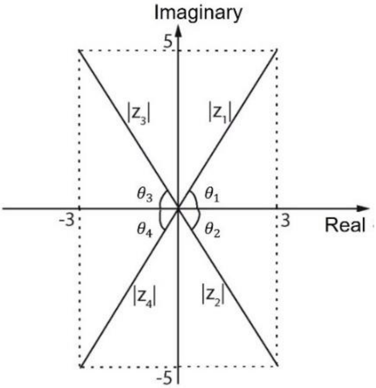

Fig. I-4. Graphical representation of $z_1 = 3 + 5i$, $z_2 = 3 - 5i$, $z_3 = -3 + 5i$, and $z_4 = -3 - 5i$.

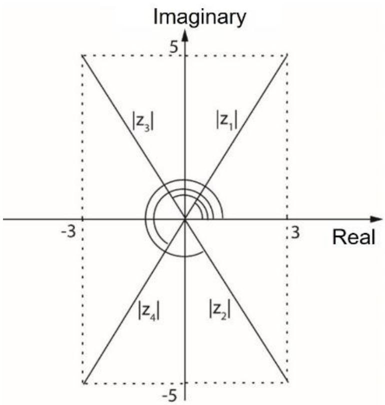

Fig. I-5. Graphical representation with the geometric arguments of $z_1 = 3 + 5i$, $z_2 = 3 - 5i$, $z_3 = -3 + 5i$, and $z_4 = -3 - 5i$. Observe the angles that start from the axis of the reals counterclockwise to the line of z.

I.4 Operations on complex numbers

Complex numbers, being numbers operated like real numbers, are considered in various operations. Below we expose the main operations with complex numbers.

Addition

The addition is done by summing the real parts together and the imaginary parts together. Let be the numbers $z_1 = a_1 + ib_1$; $z_2 = a_2 + ib_2$. Their sum gives the number z_3 such as:

$$z_3 = z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

which can be written in a compact form: $z_3 = (a_1 + a_2, b_1 + b_2)$.

Example I.4.1

1. $z_1 = 4 + i$; $z_2 = 3 + 3i$. Their sum gives $z_3 = z_1 + z_2 = (4 + 3) + i(1 + 3) = 7 + 4i$.
2. $z_1 = 4 - i$; $z_2 = 3 + 3i$. Their sum gives $z_3 = z_1 + z_2 = (4 + 3) + i\big((-1) + 3\big) = 7 + 2i$.

The graphical representation of these two examples is displayed in Fig. I-6. The sum of the real parts between them and the imaginary parts between them is equivalent to determining the total real and total imaginary components on the graph, and the number sum z_3 appears as the diagonal of the parallelogram formed by the two starting numbers z_1 and z_2.

3. $z_1 = 4 - 2i$; $z_2 = 3 + 3i$; $z_3 = 5 + 2i$; $z_4 = -6 - 5i$. Their sum gives $z_5 = z_1 + z_2 + z_3 + z_4 = (4 + 3 + 5 - 6) + i\big((-2) + 3 + 2 + (-5)\big) = 6 - 2i$.

It is not necessary from now on to write $+(-5)$ as in the previous operation, we will simply write -5.

It is difficult to represent the graph of the sum of four or more complex numbers, it would overload the figure. However, and since the addition is associative, it is possible to group numbers two by two and sum them sequentially. For example, one can obtain a number by summing z_1 and z_2 to obtain z_6, and by summing z_3 and

z_4 we obtain z_7, then we sum z_6 and z_7 to obtain z_5 as in example 3 above.

Just like real numbers, complex numbers can be located in a coordinate system based on their coordinates a and b as in $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, and we can evaluate the distance from z_1 to z_2. We have seen that the modulus of z_1 is $\sqrt{a_1^2 + b_1^2}$. This can be considered as the distance from the point $(0,0)$ to the point (a_1, b_1), either the modulus or the length. The calculation of this distance is equivalent to making the difference between the point $z_1 = a_1 + ib_1$ and the point of origin $(0,0)$, either $|z_1 - 0| = |(a_1 + ib_1) - (0 + i \times 0)| = |(a_1 - 0) + i(b_1 - 0)| = \sqrt{a_1^2 + b_1^2}$.

Now, in a similar way, consider the distance between z_1 and z_2, either $|z_1 - z_2| = |(a_1 + ib_1) - (a_2 + ib_2)| = |(a_1 - a_2) + i(b_1 - b_2)| = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$. Fig. I-7 shows numbers $z_1 = 3 + 2i$ and $z_2 = 1 + 4i$ as points in the coordinate system affected by their vectors from the origin.

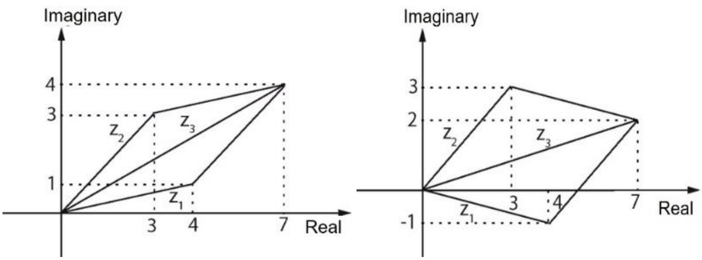

Fig. I-6. Graphical representation of (upper) $z_1 = 4 + i$, of $z_2 = 3 + 3i$ and their sum $z_3 = 7 + 4i$, and of (lower) $z_1 = 4 - i$, of $z_2 = 3 + 3i$ and their sum $z_3 = 7 + 2i$.

The distance is calculated as $d = |z_1 - z_2| = \sqrt{(3 - 1)^2 + (2 - 4)^2} = \sqrt{8}$. Also notice that the complex number $z = z_1 - z_2$ has for coordinates $z = z_1 - z_2 = (3 + 2i) - (1 + 4i) =$

$(3-1)+i(2-4)=2-2i$. The number z is found graphically as the diagonal of the parallelogram formed by z_1 and $-z_2$.

Multiplication

If the addition is made distinctly between the real numbers and the imaginary numbers, the multiplication does not make a distinction. A real number can multiply an imaginary number. Multiplication between complex numbers is done by multiplying in turn the real and imaginary parts of a number by the two real and imaginary parts of the other number.

Let be two complex numbers:

$z_1 = a_1 + ib_1;\ z_2 = a_2 + ib_2$.

The product of z_1 by z_2 is given by:

$$\begin{aligned} z_1\, z_2 &= a_1 \times (a_2 + ib_2) + ib_1 \times (a_2 + ib_2) \\ &= a_1a_2 + ia_1b_2 + ib_1a_2 + i^2b_1b_2 \\ &= a_1a_2 - b_1b_2 + i(a_1b_2 + a_2b_1) \end{aligned}$$

here we used $i^2 = -1$.

Among the multiplications of interest in complex numbers, is the multiplication of a number by its conjugate. This interest is encountered especially when there is a division by a complex number, and then we multiply the denominator by its conjugate in order to eliminate the imaginary numbers in the denominator.

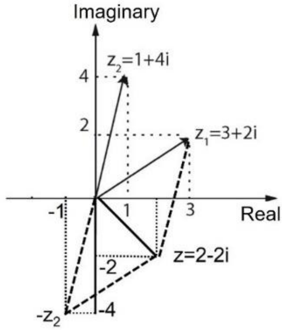

Fig. I-7. Graphical representation of $z = z_1 - z_2$. The vector $\vec{z}$ forms the diagonal between vectors $\overrightarrow{z_1}$ and $\overrightarrow{-z_2}$ indicated in discontinuous line.

Let be a complex number $z = a + ib$ where a and b are real. Its conjugate is $\bar{z} = a - ib$. Notice the indication of the conjugate by the bar above z. The multiplication of $z\bar{z} = (a + ib)(a - ib) = a(a - ib) + ib(a - ib) = a^2 - iab + iab - i^2b^2 = a^2 + b^2$.

Example I.4.2

1. $z_1 = 2 + 3i;\ z_2 = 4 + 5i;$

The multiplication of $z_1 \times z_2 = 2 \times (4 + 5i) + 3i \times (4 + 5i) = 8 + 10i + 12i + 15i^2 = -7 + 22i$.

2. $z_1 = 2 - 3i; z_2 = 4 + 5i;$

Their product $z_1 \times z_2 = 2 \times (4 + 5i) - 3i \times (4 + 5i) = 8 + 10i - 12i - 15i^2 = 23 - 2i$.

3. $z_1 = 2 + 3i; z_2 = 2 - 3i;$

We notice that z_2 is the conjugate of z_1, since they have the same real value and an opposite imaginary part. It is customary to name z_2 like z_1 with a bar: z_2=$\overline{z_1}$, and multiplication is written as $z = z_1.\overline{z_1}$.