Nonlinear Physics, from Vibration Control to Rogue Waves and Beyond

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Cambridge Scholars Publishing

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This book first published 2022

Cambridge Scholars Publishing

Lady Stephenson Library, Newcastle upon Tyne, NE6 2PA, UK

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

ISBN (10): 1-5275-8817-3
ISBN (13): 978-1-5275-8817-2

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INTRODUCTION

This book was born by a misunderstanding. Solitons, stable solitary waves, attracted scientists for their remarkable behavior. Studying this topic, researchers found that solitons are solutions of integrable nonlinear equations with astonishing mathematical properties – first of all, they are integrable by the inverse-scattering transform (IST). Many scientists ended up believing that the physics of solitons is at the heart of modern research. The 2001 Nobel Prize in physics was awarded because of many experiments on Bose-Einstein condensation, based on the nonlinear Schrodinger equation, one of the fundamental equations of soliton theory.

Now we tell you another story, solitons and coherent solutions are very important in nonlinear physics but there are also chaotic and fractal solutions we cannot neglect. The situation is complex and the asymptotic perturbation (AP) method allows us to understand some basic ideas, but we need a premise. The linear physics has been a fruitful approach for many problems, from the electromagnetism (Maxwell equation) to oscillations in engineering systems, from the harmonic oscillator to quantum mechanics. In the real world, weak nonlinear effects are always present and can change drastically the system behavior. Nowadays, nonlinear systems cannot be neglected in engineering and science. Many perturbation methods can be used to study these systems in order to predict remarkable bifurcations. In this book, we will use the AP method both for nonlinear ordinary differential equations (NODEs) and nonlinear partial differential equations (NPDEs). However, there is another approach to solve nonlinear equations – using at the beginning a few approximations in order to obtain a simple nonlinear equation we can solve explicitly (for example the Korteweg-de Vries, the sine-Gordon or the nonlinear Schrodinger (NLS) equations and so on). The main drawback of these nonlinear equations is that they are able to catch only one remarkable nonlinear effect, for instance pulse or envelope solitons. On the contrary, the AP method can be applied to the complete physical nonlinear system in order to obtain a new system of integrable equations.

In other words, the usual approach is to find exact solutions of approximate equations (KdV, sine-ordon, NLS equations and so on), while using the AP method we find approximate solutions of the exact complete physical system.

In Part I, we will study NODEs and by the AP method we will derive a suitable model system to explore the most important nonlinear system characteristics. Moreover, we illustrate two vibration control methods based on delay state feedback or nonlocal feedback. Numerical simulation confirms our method validity. A mathematical overview is necessary to understand Part II and the following chapters 5 and 6, where we are able to find new integrable equations.

The AP Method can be used in order to find approximate solutions in relevant physics problems (Part III); we will study nonlinear water waves and electron acoustic waves.

Subsequently, we turn to the nonlinear Dirac equation and show one of the most important findings of this book, i.e., the particle concept is no longer trustworthy in particle physics, because it cannot explain the quantum meaning of the new chaotic and fractal solutions that are everywhere in the nonlinear world.

We also consider the classic physics and the frequency splitting for a mass particle moving in a central field.

Chapter 11 is a gift to the Einstein-de Broglie ideas about particles as solutions of a nonlinear equation, in this case a relativistic scalar field model in 3+1 dimensions.

Moreover, we illustrate how to find new integrable nonlinear equations, likely to be of physical relevance. In the last chapters, we will study the primary resonance of the nonlinear Schrodinger equation in 2+1 dimensions and the possibility of coherent and not coherent (chaotic and/or fractal) solutions in the Born-Infeld nonlinear model of electromagnetism. This book is not exhaustive and many other research fields are available in physics, biology, sociology, economy, ecology and so on.

This course developed from a course given at the Foligno-based campus of Perugia University, Italy for students graduating in ‘Mathematical Physics’. Many teaching years allowed me to write this book and I would to thank my students at Foligno, Perugia University, Italy for their helpful and valuable suggestions.

Rome, December, 31 2021
Attilio Maccari

PART I

THE ASYMPTOTIC PERTURBATION METHOD

Introduction

In the first chapters, we describe in some detail the AP method and how it can be used to clarify the behavior of many physical systems that in the linear case become simple harmonic oscillators. We know linear physics is a successful theory and can identify many characteristics about complex systems. Nevertheless, there are always small nonlinear effects and we cannot neglect them if we want a complete picture of our system. The AP method is able to catch these nonlinear effects, various types of bifurcations and so on.

In Chapter 1, we show the basic steps of the method and then we consider a nonlinear oscillator with multiple resonant or non-resonant forcing terms. The AP method allows us to find the conditions for the quenching of the free oscillations and the conditions for its persistence. Nonlinear ordinary differential equations (NODEs), with an oscillatory behavior in the linear case, are the ideal environment for the method, and the most important finding is that a nonlinear model system can describe the oscillator characteristics.

In Chapter 2, we describe a simple extension of the method and study the parametric excitations for two internally resonant van der Pol oscillators, in the presence of a one-to-one internal resonance. We obtain the nonlinear model system for the amplitude and phase modulations in such a way that we find steady-state responses, corresponding to a periodic motion for the starting system (synchronization). Parametric excitation-response and frequency-response curves can be easily obtained. Moreover, we can perform a global analysis of the nonlinear model system and study existence and characteristics of its limit cycles. We underline that a limit cycle corresponds to a two-period amplitude and phase modulated motion for the van der Pol oscillators. We find that for very low values of the parametric excitation a two-period modulated motion is also possible and if the parametric excitation increases then the oscillation period of the modulation becomes infinite, and an infinite period bifurcation occurs.

We consider other interesting oscillators in this chapter, i.e., fractal oscillators. In this particular case, the Weierstrass function is the weak fractal forcing for the nonlinear oscillator. We observe that, being that the Weierstrass function is nowhere differentiable, we can use only suitable approximations. In the linear case, the resulting motion is simply the superposition between the fractal forcing and the standard oscillation, while in the nonlinear case, the oscillator phase and its frequency become fractal and we are able to obtain the corresponding Poincaré sections to corroborate our findings.

In Chapter 3, we introduce a new topic, i.e., active vibration control for nonlinear oscillators, and study the response of a parametrically excited van der Pol oscillator to a time delay state feedback (the control term). As usual, we obtain a nonlinear model system with two equations for the amplitude and phase modulation and we consider the effect of time delay and feedback gains from the viewpoint of vibration control and perform a global analysis of the limit cycles for the nonlinear model system, corresponding to a two-period modulated motion. In order to exclude the possibility of quasiperiodic motion and to reduce the amplitude peak of the parametric resonance, we find the appropriate choices for the feedback gains and the time delay.

In Chapter 4, we introduce a new method for vibration control and the suppression of self-excited vibrations in nonlinear oscillators. i.e., the nonlocal feedback. We consider two cases, the van der Pol equation and a nonlinear oscillator with quadratic and cubic nonlinearities. A nonlocal control force is considered in such a way to obtain a third-order nonlinear differential equation (jerk dynamics). The performance of this new control strategy is carefully considered. The feedback gains are connected with the stability and response of the system under control. Uncontrolled and controlled systems are compared and the appropriate choices for the feedback gains are found in order to reduce the amplitude peak of the self-excitations.

CHAPTER 1

NONLINEAR OSCILLATORS

1. Introduction

Linear oscillations are a fundamental topic in general physics [150]. When a system is near its equilibrium point, it begins to oscillate but if the displacement increases then the nonlinear terms are not negligible. First of all, we consider the differential equation for the harmonic oscillator

$$\frac{d^2X}{dt} + \omega^2 X(t) = 0 \qquad (1.1)$$

where X(t) is the displacement and ω the circular frequency. The most general solution is

$$X(t) = 2\rho cos(-\omega t + \theta) \qquad (1.2)$$

where ρ and θ are fixed by the initial conditions (The Cauchy problem)

$X(0) = X_0$ for the displacement and $\dot{X}(0) = \dot{X}_0$ for the initial velocity, where the dot denotes differentiation with respect to the non-dimensional time, then we easily get

$$2\rho = \sqrt{\left((X_0)^2 + \left(\frac{\dot{X}_0}{\omega}\right)^2\right)} \qquad (1.3)$$

and

$$tan\theta = \left(\frac{\dot{X}_0}{\omega X_0}\right) \qquad (1.4)$$

Now, we can consider a weakly nonlinear part in the differential equation (1.1) or on the contrary, a strongly nonlinear part but with small solutions. The first consequence is that the amplitude and the phase are slowly varying with time so we can introduce a slow time

$$\tau = \varepsilon^q t \tag{1.5}$$

where ε is a bookkeeping device and q is a rational number that will be chosen afterwards. If we want to study the asymptotic solution behavior $(t \to \infty)$ and $\varepsilon \to 0$ then τ must assume finite values. So, we assume that an approximate solution is given by

$$X(t) = 2\rho(\tau)cos\big(-\omega t + \theta(\tau)\big) = (\rho(\tau)exp(-i\omega t + i\theta) + c.c.) \tag{1.6}$$

or better

$$X(t) = \varepsilon^{(1+r)}\Psi_0 + (\varepsilon\Psi_1 exp(-i\omega t) + \varepsilon^2\Psi_2 exp(-2\omega t) + \varepsilon^3\Psi_3 exp(-3i\omega t) + c.c. + h.o.t.) \tag{1.7}$$

where c.c. stands for complex conjugate, h.o.t. for higher order terms and *r* is another rational number.

Following this path, we are mixing the most important features of two well-known perturbation methods, the harmonic balance and the multiple scale methods (for more details about these two perturbation methods [112,113, 130]).

If we consider a weakly nonlinear differential equation

$$\frac{d^2X}{dt} + \omega^2 X(t) = NL \tag{1.8}$$

where NL stands for the nonlinear part, for instance

$$aX^2(t) + bX^3(t) \tag{1.9}$$

we can insert the solution (1.7) in the nonlinear equation (1.8) and with some algebra manipulation we get for *n=0*

$$\omega^2\varepsilon^{(1+r)}\Psi_0 = 2a\varepsilon^2|\Psi|^2 \tag{1.10}$$

then r=1, for n=2

$$-3\omega^2\varepsilon^2\Psi_2 = a\varepsilon^2\Psi^2 \tag{1.11}$$

and for n=1

$$-2i\omega\varepsilon^q\psi_\tau = 2a(\varepsilon^{82} + r)\Psi_0\Psi + \varepsilon^2\Psi_2(c.c.\Psi) + 3b\varepsilon^2|\Psi|^2\Psi \tag{1.12}$$

then q=2 for the proper nonlinear terms balance and with some algebra manipulation

$$\frac{d\Psi}{d\tau} = \frac{iA}{2\omega}|\Psi|^2\Psi, \qquad A = \frac{10a^2}{3\omega^2} + b. \tag{1.13}$$

With the substitution

$$\Psi = \rho exp(i\theta) \tag{1.14}$$

where ρ is the solution amplitude and θ is its phase, we arrive at the following equations:

$$\frac{d\rho}{d\tau} = 0, \qquad \frac{d\theta}{d\tau} = \frac{A}{2\omega}\rho^2 \tag{1.15}$$

We observe that the variable change (1.5) implies that ($n \neq 0$)

$$\frac{d}{dt} \to -in\omega + \varepsilon^q \frac{d}{d\tau} \tag{1.16}$$

when the temporal differential operator acts on the function

$$\Psi_n(\tau)exp(-in\omega t) \tag{1.17}$$

From the equations (1.15), we can see that the approximate solution is always periodic, the amplitude is constant but the period changes and becomes $T = \frac{2\pi}{\Omega}$, where

$$\Omega = \omega - \frac{A}{2\omega}\rho^2 \tag{1.18}$$

However, if

$$b = -\left(\frac{10a^2}{3\omega^2}\right). \tag{1.19}$$

the period does not change and is equal to the linear case period (see (1.113)).

We now can begin our journey among other nonlinear physical systems.

2. Nonlinear dynamical systems with a finite number of harmonic forcing terms

We want to study the transient and steady-state response of a very general nonlinear oscillator subject to a finite number of harmonic forcing terms using the asymptotic perturbation (AP) method and improve previous work devoted to this topic [10, 101, 136, 137]. We consider three cases: i) the forcing frequencies are not close to each other or close to the primary resonance of the oscillator; ii) the forcing frequencies are close to each other but not close to the primary resonance; iii) all the forcing frequencies are close to the primary resonance. We determine both the conditions for the quenching of the free oscillation and the conditions for its persistence. Analytical results are validated by numerical integration.

We consider the transient and steady-state response of a nonlinear oscillator subject to a finite number of harmonic forcing terms:

$$\begin{aligned}&\ddot{X}(t) + X(t) + a\dot{X}(t) + bX^2(t) + cX(t)\dot{X}(t)\\&+d\dot{X}^2(t) + eX^3(t) + f\dot{X}(t)X^2(t)\\&+g\dot{X}^2(t)X(t) + h\dot{X}^3(t) = F(t),\end{aligned} \tag{1.20}$$

where the dots denote differentiation with respect to the non-dimensional time and *F(t)* is a finite sum of *N* harmonic forcing terms of the form

$$F(t) = \sum_{i=1}^{N} 2A_i \cos(\Omega_i t), \tag{1.21}$$

where A_i is the amplitude and Ω_i is the non-dimensional frequency of the *i*th component of *F(t)*. (All times are referred to the time scale $1/\omega$, where ω is the natural frequency of the linearized homogenous version of (1.20)).

We are primarily interested in the case N>1 and especially to the modifications induced by the nonlinear terms on the solution of the linearized version of (1.20):

$$X(t) = 2\rho_0 \cos(-t + \theta_0) + \sum_{i=1}^{N} \frac{2A_i}{(1-\Omega^2 i)} \cos(\Omega_i t) \tag{1.22}$$

where ρ_0, θ_0 are fixed by the initial conditions. This solution is the sum of the free oscillation and of the forced oscillation. In particular, it is essential to discover if the free oscillation, i.e., the first term of the r.h.s. of (1.22), will persist or decay ("quenching"), when the non-linear terms are active.

Equation (1.20) contains well-known oscillators: the van der Pol oscillator ($a, f \neq 0$ and all the others parameters zero), the Duffing oscillator ($a, e \neq 0$ and all the other parameters zero) and so on.

We will demonstrate that the most important finding is that if the forcing frequencies are not close to the primary resonant frequency, the amplitude of the oscillation will decay exponentially in time otherwise it will approach a constant value. Our purpose is to generalize these results to the general nonlinear oscillator (1.20).

It is well-known that any signal over a specific time period can be approximated by a finite number of trigonometric terms and then our study can be judiciously applied to determine the transient and steady-state response of a nonlinear oscillator subject to a virtually arbitrary signal.

The general approach is inspired by the asymptotic perturbation method [77, 78, 82, 83, 84] for dynamical systems. The formal perturbation solution is carried out to the lowest order approximation. First of all, we assume that the forcing frequencies of (1.22) are not close to each other or close to the primary resonance, but firstly we expose the most important characteristics of the asymptotic perturbation method. It comes from a similar method employed in nonlinear partial differential equations and is based on the detailed computation of the interaction, induced by the harmonics, of solutions of the linear part of the differential equation, because of the presence of the nonlinear terms.

By means of the temporal rescaling

$$\tau = e^{q} t, \qquad (1.23)$$

with q a rational positive number, which will be fixed later on, attention is devoted to the asymptotic behavior of the solution: when t®¥ and e®0, the parameter q can be chosen in such a way that t assumes finite values.

The required solution can be expressed as a perturbation expansion, based on the parameter e, which is formally written

$$X(t) = \sum_{n=-\infty}^{+\infty} e^{\gamma_n} \psi_n(\tau; \varepsilon) \exp(-\mathrm{in}\omega t) + \varepsilon \left(\sum_{i=1}^{N} \frac{A_i}{(1-\Omega^2 i)} (\exp(i\Omega_i t) + c.c.) \right) \qquad (1.24)$$

where *c. c.* stands for complex conjugate, $\gamma_n = |n|$ for $n \neq 0$, $\gamma_0 = r$ a non negative rational number, which will be fixed later on, and $\psi_n(\tau,\varepsilon) = (\tilde{\psi})_{-n}(\tau,\varepsilon)$, because X(t) is real (the tilde denotes complex conjugate).

The function $\psi_n(\tau,\varepsilon)$ depends on the parameter e and it is supposed that the limit of the ψ_n for $\varepsilon \to 0$ exists and is finite. The expansion (1.24) can be substituted into the differential equation (1.20) so as to obtain separate equations for each *n* and subsequently we equate coefficients of like powers of ε.

A key feature of the present method is that we can simultaneously take into account the advantages of the harmonic balance method (see (1.24)) and the multiple scales technique (see (1.23)). The method is constructive in a local sense, i.e., near an equilibrium point of the oscillator, so that one can reconstruct the general motion of the system.

Indicating with $\psi(\tau)$ the limit of $\psi_1(\tau,\varepsilon)$ when e®0, for n=1, the following equation is obtained:

$$-2\grave{\imath}\psi\varepsilon^{(1+q)} - ia\psi\varepsilon^3 + (2b - ic)\left(\psi_0\psi\varepsilon^{(1+r)} + \psi_2\tilde{\psi}\varepsilon^3\right) + 4d\psi_2\tilde{\psi}\varepsilon^3 + (3c - if - 3ihg)|\psi|^2\psi\varepsilon^3 + (2g\tilde{A} - 6ih\tilde{A} + 6eA - 2ifA)\psi\varepsilon^3 + h.o.t. = 0 \quad (1.25)$$

where

$$A = \sum_{i=1}^{N} \frac{A_i^2}{(1-\Omega_i^2)}, \quad \tilde{A} = \sum_{i=1}^{N} \frac{A_i^2\Omega_i^2}{(1-\Omega_i^2)} \quad (1.26)$$

and h.o.t. stands for higher order terms. For the proper balance of the various terms, we set *q=r=2*.

For n=0 and n=2, equation (1.20) yields

$$\psi_0 = -2(b+d)|\psi|^2 - 2(\mathrm{b}A + d\tilde{A}) + h.o.t. \quad (1.27)$$

$$\psi_2 = \frac{(b-d-\mathrm{ic})}{3}\psi^2 + h.o.t. \quad (1.28)$$

Equations (1.27) and (1.28) can be substituted into (1.25) and in such a way we arrive at the very nice equation

$$\psi_t = (\alpha_1 + i\alpha_2)\psi + (\beta_1 + i\beta_2)|\psi|^2\psi \quad (1.29)$$

where

$$\alpha_1 = -\frac{a}{2} + (\mathrm{bc} - f)A + (\mathrm{cd} - 3h)\bar{A} \tag{1.30}$$

$$\alpha_2 = (2b^2 - 3e)A + (2\mathrm{bd} - g)\bar{A} \tag{1.31}$$

$$\beta_1 = \frac{1}{2}\left(\mathrm{bc} + \frac{\mathrm{cd}}{3} - f - 3h\right) \tag{1.32}$$

$$\beta_2 = \frac{1}{2}\left(\frac{10}{3}(b^2 + \mathrm{bd}) + \frac{c^2+4d^2}{3} - 3c - g\right) \tag{1.33}$$

By means of the standard substitution

$$\psi(\tau) = \rho(\tau)\exp\big(i\theta(\tau)\big) \tag{1.34}$$

equation (1.29) can be separated in two parts

$$\frac{d\rho}{dt} = \alpha_1\rho + \beta_1\rho^3 \tag{1.35}$$

$$\frac{d\theta}{dt} = \alpha_2 + \beta_2\rho^2 \tag{1.36}$$

The approximate solution good to the order of ε^2 is

$$X(t) = 2\varepsilon\rho(\varepsilon^2 t)cos\big(-t + \theta(\varepsilon^2 t) - 2(b+d)\varepsilon^2\rho^2(\varepsilon^2 t) - 2(bA + d\bar{A})\big)$$

$$+\left(\frac{2}{3}\right)(b-d)\varepsilon^2\rho^2(\varepsilon^2 t)cos\big(-2t + 2\theta(\varepsilon^2 t)\big) + \varepsilon\sum_{i=1}^{N}\left(\frac{2A_i}{(1-\Omega_i^2)}cos(\Omega_i t)\right).$$

(1.37)

(1.3(1.37)7)

Note that the temporal evolution of $\rho(t)$ does not depend on $\theta(\tau)$ and then equation (1.35) can be easily integrated

$$\rho(t) = \rho_0\left[\left(1 + \frac{\beta_1\rho_0^2}{\alpha_1}\right)\exp(-2\alpha_1 t) - \frac{\beta_1\rho_0^2}{\alpha_1}\right]^{-\frac{1}{2}} \tag{1.38}$$

From inspection of (1.38), we deduce that $\rho(t)$ diverges when

$$t = t_0 = \left(\frac{1}{2\alpha_1}\right)\log\left[\frac{\beta_1\rho_0^2 + \alpha_1}{\beta_1\rho_0^2}\right] \tag{1.39}$$

if $\beta_1 > 0$, $\alpha_1 + \beta_1\rho_0^2 > 0$.

We distinguish four cases:

(i) $\alpha_1 > 0,\ \beta_1 > 0$: stable equilibrium points do not exist and the solution diverges (obviously our approximation is not valid for $t \simeq t_0$);

(ii) $\alpha_1 < 0, \beta_1 < 0$: the origin is a stable equilibrium point and $\rho(t)$ approaches zero as t goes to infinity ("quenching" of the free oscillation term in (1.37));

(iii) $\alpha_1 > 0, \beta_1 < 0$: $\rho(t)$ approaches the stable equilibrium point

$$\rho_1 = \left(-\frac{\alpha_1}{\beta_1}\right)^{\frac{1}{2}} \tag{1.40}$$

and then the free oscillation is always present. Unless the Ω_i are all rational numbers, i.e., commensurable with ω, the motion will be quasiperiodic; the asymptotic solution is

$$\begin{aligned}X(t) &= 2\rho_1 cos\big((1-\tilde{\omega})t\big) - 2(b+d)\rho_1^2 - 2(bA + d\tilde{A})\\ &+\left(\frac{2}{3}\right)(b-d)\rho_1^2 cos(2t - 2\tilde{\omega}t) + \frac{2}{3}c\rho_1^2 sin(2t - 2\tilde{\omega}t)\\ &+\sum_{i=1}^{N}\left(\frac{2A_i}{1-\Omega_i^2}\right)cos(\Omega_i t)\end{aligned} \tag{1.41}$$

where

$$\tilde{\omega} = \alpha_2 + \beta_2\rho_1^2 \tag{1.42}$$

(iv) $\alpha_1 < 0, \beta_1 > 0$: the origin is a stable equilibrium point ("quenching" of the oscillation) and (1.40) is unstable. If $\rho_0 > \rho_1$ then the solution diverges when $t \simeq t_0$.

Numerical integration of (1.20) confirms the qualitative picture which emerges from our analysis. For example, in Figure 1, we show the numerical solution compared with our approximate solution (1.41) for the case iii). The mean difference between the two solutions is 0.002, i.e., of order ε^3 as expected.

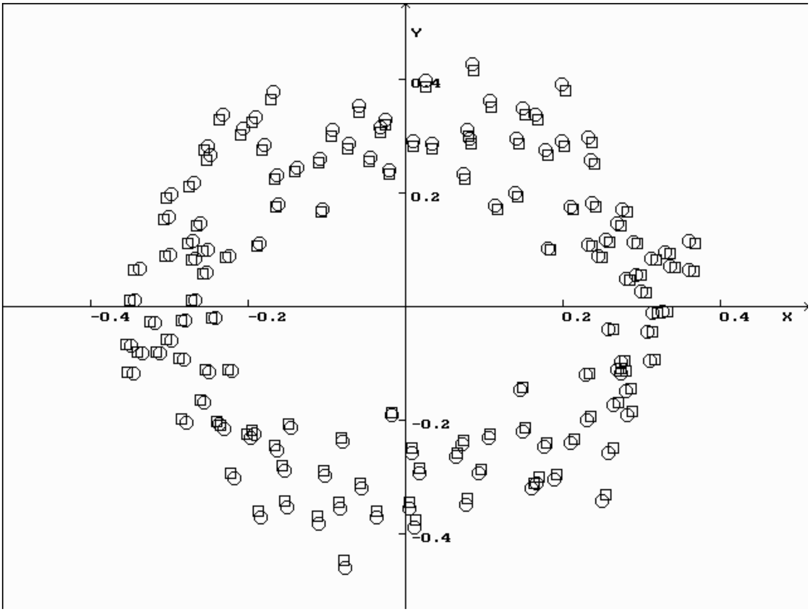

Figure 1: Comparison between numerical (rectangles) and analytical (circles) solutions in the $(X, \dot{X} = Y)$ plane.

Values of parameters: *a=-0.01, b=1.5, c=-0.9, d=0.2, e=1.0, f=0.1, g=-0.1, h=-0.3.*

Forcing frequencies not close to each other and not close to the primary resonance: $\Omega_1 = \sqrt{3}$, $\Omega_2 = \sqrt{5}$, $\Omega_3 = \sqrt{7}$

Amplitudes of the external excitations: $A_1 = 0.03$, $A_2 = 0.05$, $A_3 = 0.05$.

3. The approximate solution with the frequencies close to each other

The results of the previous section can be extended to the case when the forcing frequencies are close to each other, but not close to the primary frequency.

We let

$$\Omega_i = \Omega + \varepsilon^2\sigma_i, \quad i = 1....N \tag{1.43}$$

where Ω is a fixed frequency not close to one, while σ_i measures the differences of the frequencies from each other. Substituting (1.43) in (1.21), we find that *F(t)* becomes

$$F(t) = \frac{\exp(i\Omega t)}{(1-\Omega^2)}\sum_{i=1}^{N} \varepsilon A_i \exp(i\sigma_i t) + c.c. + O(\varepsilon^3) \tag{1.44}$$

The non-linear oscillator is then subject to an applied force with frequency Ω and with an amplitude that is a slowly varying function of time.

We apply the same method as in section 2 and obtain the equations (1.35-1.36) but now with

$$\alpha_1(t)\text{=-}\frac{a}{2} + (\mathrm{bc} + \mathrm{cd}\Omega^2 - 3h\Omega^2 - f)A(t) \tag{1.45}$$

$$\alpha_2(t) = (2b^2 + 2\mathrm{bd}\Omega^2 - g\Omega^2 - 3e)A(t) \tag{1.46}$$

where

$$A(t) = \frac{1}{(1-\Omega^2)}\sum_{i=1}^{N} A_i^2 + \frac{1}{(1-\Omega^2)}\sum_{i,j=1(i\neq j)}^{N} A_i A_j \exp\big(i(\sigma_i - \sigma_j)t\big), \tag{1.47}$$

and β_1, β_2 unchanged. Also, in this case the evolution of $\rho(t)$ does not depend on $\theta(t)$, but the difference is now that α_1 and α_2 are explicitly dependent on t. It is advantageous to introduce a new function $\gamma(t)$, with the following characteristics:

$$\dot{\gamma}(t) = \alpha_1(t), \quad \gamma(0) = 0 \tag{1.48}$$

where the dot denotes differentiation with respect to t.

A simple integration shows that

$$\gamma(t)\text{=-}\frac{a}{2}t + (\mathrm{bc} + \mathrm{cd}\Omega^2 - 3h\Omega^2 - f)\begin{pmatrix} \dfrac{t}{(1-\Omega^2)}\displaystyle\sum_{i=1}^{N} A_i^2 - \dfrac{i}{(1-\Omega^2)} \\ \displaystyle\sum_{i,j=1(i\neq j)}^{N} A_i A_j \dfrac{\big(\exp\big(i(\sigma_i-\sigma_j)t\big)-1\big)}{(\sigma_i-\sigma_j)} \end{pmatrix} \tag{1.49}$$

This function can be separated into two parts

$$\gamma(t) = Bt + \delta(t) \tag{1.50}$$

where

$$B = -\frac{a}{2} + (\mathrm{bc} + \mathrm{cd}\Omega^2 - 3h\Omega^2 - f)\frac{1}{(1-\Omega^2)}\sum_{i=1}^{N} A_i^2 \tag{1.51}$$

and $d(t)$indicates the oscillating part. The temporal evolution of $\rho(t)$ is now

$$\rho(t) = \frac{\rho_0 \exp(\gamma(t))}{\sqrt{\left|1-2\beta_1\rho_0^2\int_0^t \exp(2\gamma(t\prime))dt\prime\right|}} \tag{1.52}$$

The behavior of ρ as t becomes large can be easily determined if we consider $\gamma(t)$ in the form (1.50). It is straightforward to show that if B<0, the asymptotic behavior is

$$\rho(t) \sim \exp(Bt), \text{ as } t \to \infty \text{ (B<0)} \tag{1.53}$$

and then we again obtain the decay of the free oscillation and the quenching of the solution. If B>0 a new behavior arises, not observable for *N=1*: the amplitude of the free oscillation approaches an oscillatory function of time, which depends on both the amplitudes A_i as well as the detuning parameters σ_i:

$$\rho(t) \sim exp\left(\left(\delta(t)\right)\right), \text{ as } t \to \infty \text{ (B>0)} \tag{1.54}$$

In Figure 2, we show the numerical solution compared with our approximation. The mean difference between the two solutions is 0.004, i.e., of order ε^3 as expected.

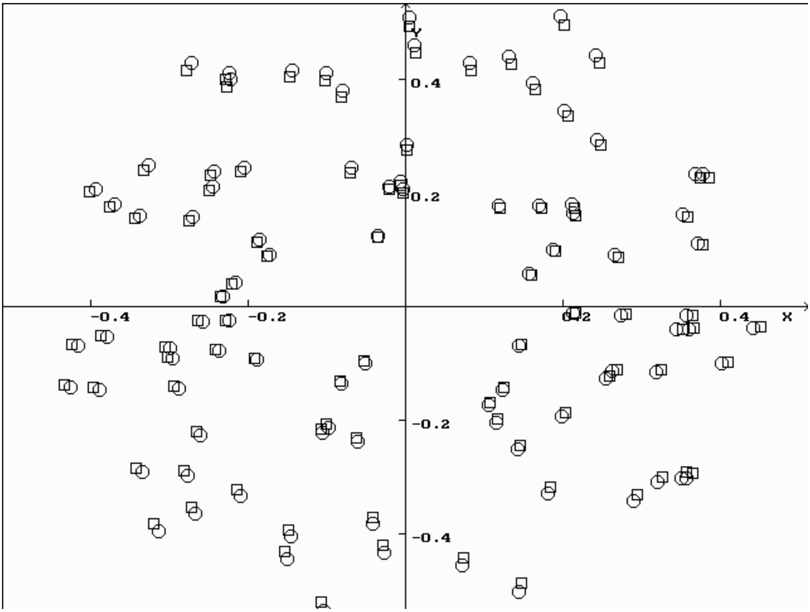

Figure 2: Comparison between numerical (rectangles) and analytical (circles) solutions in the $(X, \dot{X} = Y)$ plane.

Values of parameters: a=-0.01, b=1.5, c=-0.9, d=0.2, e=1.0, f=0.1, g=-0.1, h=-0.3.

Forcing frequencies close to each other but not close to the primary resonance:

$\Omega_1 = \sqrt{3}$, $\Omega_2 = \sqrt{3.1}$, $\Omega_3 = \sqrt{3.2}$

Amplitudes of the external excitations: $A_1 = 0.03$, $A_2 = 0.05$, $A_3 = 0.05$.

4. Forcing frequencies near primary resonance

We consider the case when the frequency of each component of the forcing term is near the primary resonant frequency of the oscillator. We let

$$\Omega_i = 1 + \varepsilon^2 \sigma_i, \quad i = 1....N \tag{1.55}$$

where σ_i measures the differences of the frequencies from the natural frequency of the oscillator. Substituting (1.55) in (1.21), we find that *F(t)* becomes

$$F(t) = \varepsilon^3 \mathrm{exp(it)} \sum_{i=1}^{N} A_i \mathrm{exp}(i\sigma_i \tau) + c.c. \tag{1.56}$$

The non-linear oscillator is then subject to an applied force with N different frequencies and amplitudes, which are supposed to be of order ε^3, because we are in the primary resonance zone.

We now look for a solution in the form

$$X(t) = \sum_{n=-\infty}^{+\infty} \varepsilon^{\gamma_n} \psi_n(\tau;\varepsilon) \mathrm{exp}(-\mathrm{in}\omega t) \tag{1.57}$$

with the same conventions as in (1.24).

We substitute (1.57) in (1.20) so as to obtain different equations for each n and subsequently we equate coefficients of like powers of ε to obtain

$$-2i\psi_\tau - \mathrm{ia}\psi + (2b - \mathrm{ic})(\psi_0\psi + \psi_2\bar{\psi}) + 4d\psi_2\bar{\psi} + \\ (3c - \mathrm{if} - 3\mathrm{ih} + g)|\psi|^2\psi - \sum_{i=1}^{N} A_i \mathrm{exp}(-i\sigma_i t) = 0 \tag{1.58}$$

We do not give the details of the calculation and furnish the final results. By means of the substitution (1.34), we obtain the equations for the amplitude and the phase of the free oscillation

$$\frac{d\rho}{d\tau} = \alpha_1\rho + \beta_1\rho^3 + \frac{1}{2}\sum_{i=1}^{N} A_i \sin(\sigma_i t - \theta) \tag{1.59}$$

$$\rho\frac{d\theta}{d\tau} = \beta_2\rho^3 + \frac{1}{2}\sum_{i=1}^{N} A_i \cos(\sigma_i \tau - \theta) \tag{1.60}$$

where

$$\alpha_1 = -\frac{a}{2}, \tag{1.61}$$

and β_1 , β_2 are given by (1.32-1.33).

The difference with the preceding cases is that now equations (1.59-1.60) are two coupled nonlinear differential equations, which must be integrated numerically.

However, a very interesting behavior is observed if $\alpha_1 > 0$ and $\beta_1 < 0$ and

$$\beta_2\rho_1^3 >> \sum_{i=1}^{N}|A_i|, \quad \text{with } \rho_1 = \left(-\frac{\alpha_1}{\beta_1}\right)^{\frac{1}{2}} \tag{1.62}$$

i.e., for weak external excitations. In this case, at least for initial conditions near ρ_1, the system (40-41) can be approximated by

$$\frac{d\rho}{dt} = -2\alpha_1\rho + \frac{1}{2}\sum_{i=1}^{N} A_i \sin(\sigma_i t + \Omega t + \theta_0) \tag{1.63}$$

$$\theta = \Omega\tau + \theta_0 \tag{1.64}$$

where $\Omega = \beta_2\rho_1^2$. The solution of (1.63) is

$$\rho(t) = \rho_0 exp(-2\alpha_1 t) + \sum_{i=1}^{N} \frac{A_i}{2(4\alpha_1^2 + \Omega_i^2)}(2\alpha_1 sin(\Omega_i t + \theta_0) - \Omega_i cos(\Omega_i t + \theta_0) + AD) \tag{1.65}$$

where *AD* is

$$AD = exp(-2\alpha_1 t)(\Omega_i cos\theta_0 - 2\alpha_1 sin\theta_0) \tag{1.66}$$

$$\Omega_i = \Omega + \sigma_i \tag{1.67}$$

The asymptotic behavior of (1.65) is

$$\rho(t) = \sum_{i=1}^{N} \frac{A_i}{2(4\alpha_1^2 + \Omega_i^2)}\left(2\alpha_1 sin(\Omega_i t + \theta_0) - \Omega_i cos(\Omega_i t + \theta_0)\right) \tag{1.68}$$

The approximate solution good to the order of ε^2 is

$$X(t) = 2\rho(t)cos\big((1-\Omega)t + \theta_0\big) - 2(b+d)\rho^2(t) - 2(bA + d\bar{A}) + \left(\frac{2}{3}\right)(b-d)\rho^2(t)cos(2t - 2\Omega t + \theta_0) + \frac{2}{3}c\rho^2(t)sin(2t - 2\Omega t + 2\theta_0) \tag{1.69}$$

where $\rho(t)$ is given by (1.68).

In Figure 3, we show a comparison between the numerical solution of (1.20) and the approximate solution (1.68). The mean difference between the two solutions is 0.003, i.e., of order ε^3 as expected.

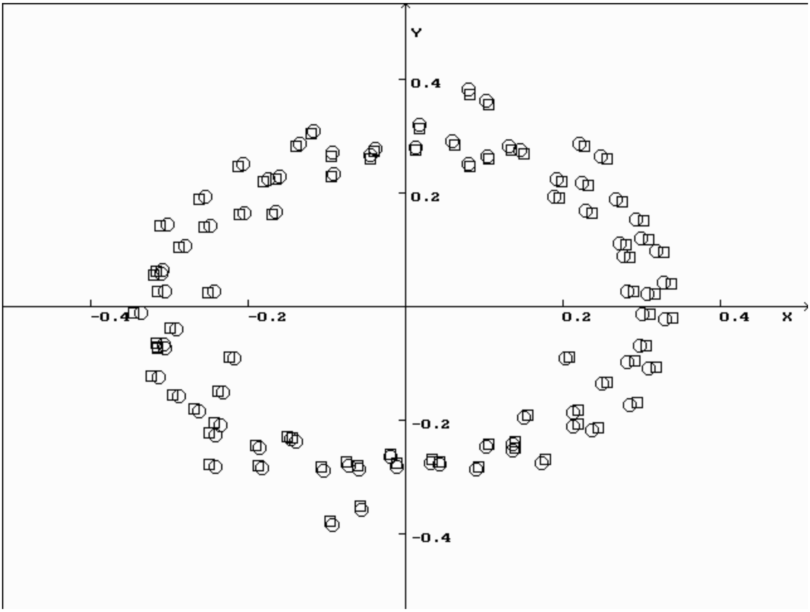

Figure 3: Comparison between numerical (rectangles) and analytical (circles) solutions in the $(X, \dot{X} = Y)$ plane.

Values of parameters: *a=-0.01, b=1.2, c=-0.9, d=0, e=1.2, f=0.3, g=-0.2, h=-0.3.*

Forcing frequencies close to each other and close to the primary resonance:

$$\Omega_1 = \sqrt{1.3}, \quad \Omega_2 = \sqrt{1.2}, \quad \Omega_3 = \sqrt{1.1}$$

Amplitudes of the external excitations: $A_1 = 0.003$, $A_2 = 0.002$, $A_3 = 0.002$.

5. Conclusion

We have used the asymptotic perturbation method to analyze the transient and steady-state response of a very general nonlinear oscillator under a finite number of harmonic forcing terms. Three cases of different forcing frequencies are investigated and the corresponding analytical results are compared to numerical simulations. If the forcing frequencies are not close to each other or close to the resonant frequency, then the original free oscillation can vanish (“quenching”) or maintain a finite value.

When the forcing frequencies are all close to a particular frequency Ω, the “quenching” is possible but in certain cases the amplitude of the free oscillation oscillates with a frequency determined by the detuning parameters.

When the forcing frequencies are close to the resonant frequency, then both the amplitude and the phase of the free oscillation can eventually oscillate with a frequency that is determined by both the forcing amplitudes A_i and the detuning parameters σ_i.

If we want to calculate the second order approximation solution, the amount and complexity of the algebraic computations required increase in a very dramatic manner. Consequently, the use of symbolic manipulation systems is strongly recommended.

The problem considered in this chapter clearly demonstrates the power of the asymptotic perturbation method. An important feature of this method is that it provides quantitative results regarding dynamic behavior, in contrast to much of the current work in dynamical systems theory, which is concerned with qualitative behavior.

CHAPTER 2

BIFURCATIONS AND NONLINEAR OSCILLATORS

1. Introduction

Approximate analytical and numerical methods have been applied to the van der Pol oscillator, and its dynamics have been studied in detail over the last years. Now we want to turn to the investigation of the behavior of two nonlinearly coupled van der Pol oscillators under the effect of a parametric excitation and an internal resonance. The response of a system of two nonlinearly coupled van der Pol oscillators to a principal parametric excitation in the presence of one-to-one internal resonance is investigated. The asymptotic perturbation (AP) method is applied to derive the slow flow equations governing the modulation of the amplitudes and the phases of the two oscillators. These equations are used to determine steady state responses, corresponding to a periodic motion for the starting system (synchronization), and parametric excitation-response and frequency-response curves. Energy considerations are used to study existence and characteristics of limit cycles of the slow flow equations. A limit cycle corresponds to a two-period amplitude- and phase-modulated motion for the van der Pol oscillators. Two-period modulated motion is also possible for very low values of the parametric excitation and an approximate analytic solution is constructed for this case. If the parametric excitation increases, the oscillation period of the modulations becomes infinite and an infinite-period bifurcations occur. Analytical results are checked with numerical simulations. Therefore, we study a class of nonlinear systems described by the equations

$$\ddot{X} + \left(\omega_1^2 - 2\varepsilon f \cos(\Omega t)\right)X - \varepsilon\left(1 - X^2 - \mathrm{a}\mathrm{Y}^2\right)\dot{X} = 0, \tag{2.1}$$

$$\ddot{Y} + \omega_2^2 Y - \varepsilon\left(1 - \mathrm{b}\mathrm{X}^2 - Y^2\right)\dot{Y} = 0, \tag{2.2}$$

where dot denotes differentiation with respect to time, $\omega_1 \square \omega_2$ are the fundamental frequencies (internal resonance *1:1*), the parametric excitation frequency is Ω»$2\omega_1$, the constants a, b are of order *1*, and □ is a small parameter.

Two-degree-of-freedoms and multi-degree-of-freedoms systems under the action of parametric excitations have been extensively studied. The most important feature in these systems is a nonlinear interaction between parametric and self-excitations. Usually, an entrainment of vibration occurs and a synchronization phenomenon is observed, i.e., the system behavior is characterized by a simple periodic motion [112, 113]. For example, Asmis and Tso [8] considered the response of two-degree-of-freedom systems with cubic nonlinearities to a combination parametric resonance of the sum type in the presence of one-to-one internal resonance. Tso and Asmis [138] analyzed the response of two-degree-of-freedom systems with cubic nonlinearities and no internal resonances to a parametric harmonic excitation. The absence of Hopf bifurcations was demonstrated by Miles [109] for an internally resonant pendulum with the lower mode excited by a principal parametric excitation. Internally resonant two-degree-of-freedom systems with quadratic nonlinearities and combination parametric resonance were considered by Nayfeh and Zavodney [115]. Using numerical integration they demonstrated limit-cycle behavior and modulated response in the amplitude and phase of the oscillation. Asrar [9] used the method of multiple scales for a system with quadratic nonlinearities in the case of a principal parametric resonance and a three-to-one internal resonance and derived stability conditions for the steady-state solutions.

Nayfeh and Chin [114] examined the response of a parametrically system with cubic nonlinearities and widely spaced frequencies. In some cases, energy can be transferred from high- to low-frequency modes and chaotic response coexist with periodic behavior. Yu and Huseyin [141] performed a theoretical study about parametrically excited systems and compared the Chen-Langford and the harmonic balance methods. They found that the two methods furnish qualitatively equivalent results.

Warminski, Litak and Szabelski [142] have analyzed synchronization and chaos in a parametrically and self-excited system with two degrees of freedom. The system is formed by two van der Pol oscillators coupled by a linear spring with a periodically changing stiffness of the Mathieu type. Existence and stability of periodic solutions is investigated and regions of chaotic response are found.