Introduction to General Relativity, Cosmology and Alternative Theories of Gravitation

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By

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Dedicated to my parents and to my Guru Dr. M. N. Mahanta
who introduced me to this great subject
—D.R.K. Reddy

CONTENTS

PREFACE

This book provides a thorough introduction to Einstein's special and general relativity theories and modified theories of gravitation and their applications to cosmology. It is intended for undergraduate and postgraduate students of Physics and Applied Mathematics and is also helpful as a reference book for general relativity and cosmology researchers. This book is designed so that a student with little knowledge of relativity theories but with a sound mathematical background can understand the new developments in the subject and can effectively pursue advanced research. Tensor methods, Einstein's special and general relativity theories, cosmological models of the universe, and the alternative theories of gravitation presented here are fascinating and famous topics for any mathematician or physicist.

The book comprises five chapters. Chapter 1 contains some essential concepts of the special theory of relativity, which are necessary for further understanding general relativity and cosmology. Special relativity takes care of uniform translatory motion in a region of free space where gravitational effects are neglected. Hence a detailed discussion of it has not been taken up in this book. In chapter 2, we have incorporated the basic concepts and main results of the tensor analysis, which play a vital role in formulating Einstein's general theory of relativity, which explains the relativity of all kinds of motion and takes care of all kinds of motion gravitational effects of four-dimensional space-time. The results presented will help provide a thorough understanding of general relativity and cosmology. Chapter 3 gives a lucid introduction to the concepts of general relativity and the derivation of Einstein's field equations and their applications. It also discusses the derivation of the Schwarzschild and Reissner-Nordstrom solutions. In chapter 4, a detailed discussion of cosmology and the cosmological models of the universe and their physical properties has been given. We have also presented a special discussion of anisotropic cosmological models, which play a significant role in discussing the early stages of the evolution of the universe. Finally, in chapter 5, we have included a brief discussion of alternative or modified theories of gravitation, namely the Brans-Dicke and Saez-Ballester scalar-tensor theories of gravitation and modified theories like $f(R)$ and $f(R,T)$ (R is Ricci tensor and T is the trace of the energy-momentum tensor of matter) which have been the subjects of recent investigations in modern cosmology. These last two chapters are very much rooted in the current research work that is going on around the globe.

The concepts in the book have been explained explicitly, and the presentation of the material is so that either the beginner or the expert of the subject can easily understand and appreciate what is being discussed. This book also gives an introduction to and motivation for every topic covered in the textbook. Many exercises and solved examples have been included in each chapter to enable students to understand the ideas and derivations more easily. We are confident that this book will be helpful to students and will be appreciated by the scholars of this great subject.

It may be said that writing a textbook and bringing it into this final form is an excellent but strenuous job. In this task, several well-wishers have helped us directly or indirectly.

At the outset, we are grateful to Helen Edwards, the commissioning editor, and her efficient team at Cambridge Scholars Publishing, who were kind enough to bring this book into this beautiful final form.

D.R.K. Reddy finds pleasure in expressing his love to his wife D. Saraswathi and family members who have showered pleasant facilities while writing this book. He cannot forget the inspiration given to him by his great friend Sanagala Srinivas and his family members. He is also greatly indebted to the Almighty, who has given him health and strength while writing this book. He is also thankful to his students Dr. V. Uma Maheswara Rao and Dr. R. Lakshun Naidu, for their support. Last but not least, he is grateful to Dr. T. M. Karade and many other relativity friends of Nagpur and Amravathi for their interest.

Y. Aditya owes an enormous debt of gratitude to his mentor D.R.K. Reddy for not giving up on his dream of writing a textbook on Relativity and Cosmology and making him a part of his dream project. He is very much thankful to his Gurus, Prof. V. Uma Maheswara Rao, and Prof. M. Vijaya Santhi, for their support. He is grateful to his awesome wife, Dr. U.Y. Divya Prasanthi, for inspiring him and helping with the manuscript's LaTeX typing. Most importantly, he owes a great debt of gratitude to his beloved parents and family members for their continuous love and support. Last but not least, he is highly thankful to Dr. J. Girish, Director Education, GMR Varalaxmi Foundation, and Prof. CLVRSV Prasad, Principal and the management of GMR Institute of Technology, Rajam, for their cooperation and support.

Healthy criticism and constructive suggestions to improve the quality of the book are always welcome.

D.R.K. Reddy
Y. Aditya

CHAPTER 1

THE SPECIAL THEORY OF RELATIVITY AND ITS CONSEQUENCES

1.1 General Introduction

It is well known that the fundamental concept in the physical world is the motion of bodies, and either applied forces or central forces cause it. Classical mechanics was developed based on the notion of absolute motion governed by Newton's three laws of motion. Subsequently, the null result of the Michelson - Morley experiment ruled out the possibility of absolute motion. At this juncture, Einstein entered the scene and formulated his special theory of relativity in 1905. However, it has been said that the special theory of relativity originated not in certain thoughts in Einstein's mind when he was very young, but was due to Poincare and Lorentz.

This book aims not to go into these arguments but to present some basic principles of the special theory of relativity and briefly mention the significant physical consequences of the theory. It is now a fact that this theory has thoroughly revolutionized our ideas about physics in general and classical mechanics in particular.

This chapter is mainly devoted to a brief discussion of the postulates of the special theory of relativity, their applications, and consequences in classical mechanics and electromagnetic phenomena. Detailed discussions and derivations are not considered. Thus, this chapter provides a bird's eye view of the special theory of relativity.

1.2 Postulates of the Special Theory of Relativity

This section presents the basic principles or postulates of the special theory of relativity and its consequences.

Postulates:
If S and S' are two inertial systems (Fig. 1-1, coordinate systems which are in uniform translatory motion in which Newton's first law is valid), then

i. *The principle of relativity*: All the laws of physics are identical in all inertial systems.
ii. *The speed of light is invariant*: The velocity of light is constant in empty space and is independent of the motion and position of the observer and light source. The velocity of light is usually denoted by c and is equal to $3 \times 10^{10} cms/sec$.
iii. *Uniform motion is invariant*: A particle at rest or in constant velocity in one inertial frame will be at rest or in constant velocity in all inertial frames.

Remark: It may be noted that the above postulates are valid for uniform translatory motions only. However, in nature, this concept of motion is only hypothetical. The effects of gravity are not considered here.

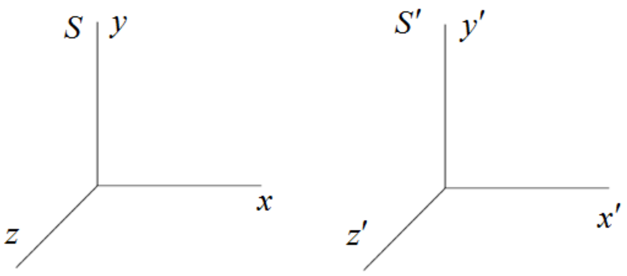

Figure 1-1: Two frames in standard configuration. Frame S' moves at velocity v relative to the frame S along the x-axis.

Consequences:

The immediate consequence of the special theory of relativity is the special Lorentz transformations. Using the Lorentz transformations, we write down the other consequences.

i. *Lorentz transformation equations:* If S and S' are two systems in uniform, relative and translatory motion and if x, y, z, t and x', y', z', t' are the coordinates of the observer in the two systems (Fig. 1-1), then

$$x' = \beta(x - vt),\ y' = y,\ z' = z,\ \text{and}\ t' = \beta\left(t - \frac{xv}{c^2}\right) \tag{1.1}$$

where v is the uniform relative velocity of two systems, $\beta = \left(1 - \frac{v^2}{c^2}\right)^{-1}$ and c is the velocity of light.

If we have two point events, each of them can be represented by its space-time coordinates in S and /or S'. For one space dimension, we can draw an $x - t$ graph and mark the individual point events as in Fig. 1-1. We have:

Event 1:

$$x_1' = \beta(x_1 - vt_1);\quad x_1 = \beta(x_1' + vt_1')$$

$$t_1' = \beta\left(t_1 - v\frac{x_1}{c^2}\right);\quad t_1 = \beta(t_1' + v\frac{x_1'}{c^2}).$$

Event 2:

$$x_2' = \beta(x_2 - vt_2);\quad x_2 = \beta(x_2' + vt_2')$$

$$t_2' = \beta\left(t_2 - v\frac{x_2}{c^2}\right);\quad t_2 = \beta(t_2' + v\frac{x_2'}{c^2}).$$

We can then evaluate the separation of the events in space and time in either frame. Thus

$$x_2' - x_1' = \beta[(x_2 - x_1) - v(t_2 - t_1)]$$

$$t_2' - t_1' = \beta\left[(t_2 - t_1) - \frac{v(x_2 - x_1)}{c^2}\right]. \tag{1.2}$$

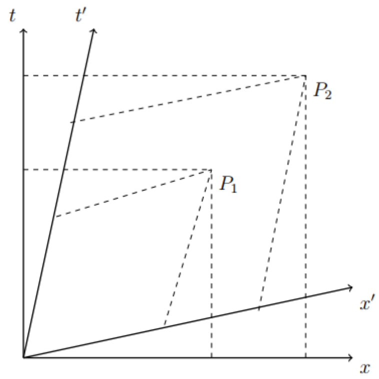

Figure 1-2: Two different point events, each describable in either S or S'.

Example 1-1 *Frame S' has a speed $v = 0.6\,c$ relative to S. Clocks are adjusted so that $t = t' = 0$ at $x = x' = 0$. Two events occur. Event 1 occurs at $x_1 = 10\,m$, $t_1 = 2 \times 10^{-7}$ sec. Event 2 occurs at $x_2 = 50\,m$, $t_2 = 3 \times 10^{-7}$ sec. What is the distance between the events and the time difference as measured in S'?*

Sol.: First, we have

$$\frac{v^2}{c^2} = \frac{9}{25}$$

and hence

$$\beta = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \frac{5}{4}.$$

Then we have

$$x_2' - x_1' = \beta[(x_2 - x_1) - v(t_2 - t_1)]$$
$$= \frac{5}{4}[(50 - 10) - \frac{3}{5}(3 \times 10^8)(3 - 2)10^{-7}]$$
$$= 27.5\,m.$$
$$t_2' - t_1' = \beta\left[(t_2 - t_1) - \frac{v(x_2 - x_1)}{c^2}\right]$$
$$= \frac{5}{4}\left[(3 - 2)10^{-7} - \frac{3}{5}\left(\frac{50 - 10}{3 \times 10^8}\right)\right]$$
$$= 2.5 \times 10^{-8} sec.$$

Remark: Eqns. (1.1), usually known as the special Lorentz transformation equations, can be uniquely derived using the two postulates of relativity and assuming Euclidean geometry and the homogeneity of space and time. If the relative velocity between the systems is small compared to that of light, then we obtain the well-known Galilean transformation equations

$$x' = x - vt,\ y' = y,\ z' = z,\ t' = t \tag{1.3}$$

It may also be remarked that the set of all Lorentz transformations form a group.

Consequences of Lorentz Transformation equations:

With the help of the Lorentz transformation equations given by Eqn. (1.1) we can obtain the following:

i. **Length contraction:** If l and l' are lengths of rods measured by two observers in relative uniform translatory motion in two systems S and S', then we have

$$l' = l\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} < l \tag{1.4}$$

$$\therefore l' < l$$

This contraction is usually called an apparent Lorentz-Fitzgerald length contraction.

ii. **Time dilation:** If S and S' are two systems in uniform, relative and translatory motion, then the time interval between two events is given by

$$dt' = \beta dt = \frac{dt}{\sqrt{1-\frac{v^2}{c^2}}} \tag{1.5}$$

where dt and dt' are the time intervals between two events measured in the systems S and S' respectively.

Transformation Equations for Velocity:

Using the Lorentz transformations, we can transform the velocities measured from one system of coordinates to the other by the following formulae:

$$
\begin{aligned}
u_{x'} &= \frac{u_x - v}{1-\frac{u_x v}{c^2}}, \\
u_{y'} &= \frac{u_y\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{u_x v}{c^2}} \\
u_{z'} &= \frac{u_z\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{u_x v}{c^2}}
\end{aligned}
\tag{1.6}
$$

where $u_x = \frac{dx}{dt}$, $u_y = \frac{dy}{dt}$, $u_z = \frac{dz}{dt}$ and u_x, u_y, u_z and $u_{x'}$, $u_{y'}$, $u_{z'}$ are the velocity components in the systems S and S' respectively.

Remark: From Eqn. (1.6) we obtain the reciprocal equations for the transformation as

$$
\begin{aligned}
u_x &= \frac{u_{x'} + v}{1+\frac{u_{x'} v}{c^2}}, \\
u_y &= \frac{u_{y'}\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{u_{x'} v}{c^2}} \\
u_z &= \frac{u_{z'}\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{u_{x'} v}{c^2}}
\end{aligned}
\tag{1.7}
$$

Note: If we put $u_{x'} = c$, $v = c$, we obtain from Eqn. (1.7) $u_x = \frac{c+c}{1+\frac{c^2}{c^2}} = c$, which shows that the velocity of light is considered to be an upper limit.

Composition of velocities:

Imagine three frames of reference in the standard configuration. S' the frame moves with velocity v_1 with respect to (w.r.t) frame S, and frame S'' moves with velocity v_2 w.r.t. frame S'. Newtonian physics gives us that frame S'' moves with velocity $v_3 = v_1 + v_2$ w.r.t. frame S, a simple velocity addition law. However, this relation does not hold if the velocities are a significant light fraction of light's speed. To get the correct relation, we simply construct two Lorentz transformations.

Example 1-2 *Derive the relativistic velocity composition law.*

Solution: Using $\beta = \frac{v}{c}$, the matrix representation of a Lorentz transformation between the frames S and S' is

$$L_1 = \begin{bmatrix} \frac{1}{\sqrt{1-\beta_1^2}} & \frac{-\beta_1}{\sqrt{1-\beta_1^2}} & 0 & 0 \\ \frac{-\beta_1}{\sqrt{1-\beta_1^2}} & \frac{1}{\sqrt{1-\beta_1^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Similarly, the transformation between S' and S'' is

$$L_2 = \begin{bmatrix} \frac{1}{\sqrt{1-\beta_2^2}} & \frac{-\beta_2}{\sqrt{1-\beta_2^2}} & 0 & 0 \\ \frac{-\beta_2}{\sqrt{1-\beta_2^2}} & \frac{1}{\sqrt{1-\beta_2^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now, we can get the Lorentz transformation between frames S and S'' by computing the product of L_2 and L_1 which are given above. We find

$$L_2L_1 = \begin{bmatrix} \frac{1+\beta_1\beta_2}{(\sqrt{1-\beta_2^2})(\sqrt{1-\beta_2^2})} & \frac{-(\beta_1+\beta_2)}{(\sqrt{1-\beta_2^2})(\sqrt{1-\beta_2^2})} & 0 & 0 \\ \frac{-(\beta_1+\beta_2)}{(\sqrt{1-\beta_2^2})(\sqrt{1-\beta_2^2})} & \frac{1+\beta_1\beta_2}{(\sqrt{1-\beta_2^2})(\sqrt{1-\beta_2^2})} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = L_3 \ (say).$$

The above matrix is itself a Lorentz transformation, and so it must have the form

$$L_3 = \begin{bmatrix} \frac{1}{\sqrt{1-\beta_3^2}} & \frac{-\beta_3}{\sqrt{1-\beta_3^2}} & 0 & 0 \\ \frac{-\beta_3}{\sqrt{1-\beta_3^2}} & \frac{1}{\sqrt{1-\beta_3^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We can find the value of β_3 by equating the related terms. Let us pick the terms in the upper left corner of each matrix

$$\frac{1}{\sqrt{1-\beta_3^2}} = \frac{1+\beta_1\beta_2}{(\sqrt{1-\beta_2^2})(\sqrt{1-\beta_2^2})} \tag{1.8}$$

After some simple mathematical calculations, we get

$$\beta_3 = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2} \tag{1.9}$$

Given $\beta = \frac{v}{c}$, from Eqn. (1.9), we get

$$v_3 = \frac{v_1 + v_2}{1 + (v_1 v_2)/c^2} \tag{1.10}$$

1.3 Four-Dimensional Space-Time Continuum - Minkowski Space

Before we move into the presentation of the consequences of special relativity in mechanics it is worthwhile to discuss the concept of the four-dimensional space-time continuum and Minkowski space. It is evident from the consequences of the special theory of relativity that the spatial and temporal (time) coordinates (measurements) are intimately connected. This leads us to believe that we no longer live in the world of points but in a world of events characterized by the coordinates (x, y, z, t), representing the location of a single point in the four-dimensional continuum. The space-time represented is usually called Minkowski space.

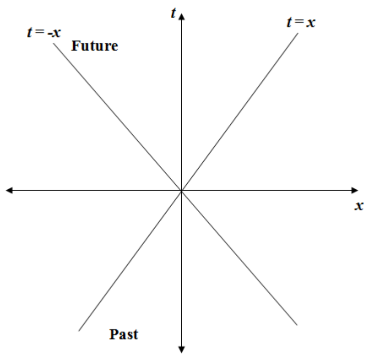

Figure 1-3: The division of space-time into past and future regions. Light rays move on both lines $t = \pm x$. These lines give the light cone and the origin is some event E in space-time. The inside region of the lower half of the light cone is the past of E, where we find all events in the past that could affect E. Inside, the light cone defined in the upper half-plane represents the future of E, and these events are affected by E. Regions outside the light cone are called space-like.

In this continuum, spatial coordinates can be represented by three mutually perpendicular axes. In contrast, the temporal coordinate cannot be shown on

the real axis; it remains as an imaginary coordinate and the time axis becomes imaginary.
Hence the interval between two events in Minkowski space is given by

$$ds^2 = dx^2 + dy^2 + dz^2 + (icdt)^2$$
$$i.e., \; ds^2 = dx^2 + dy^2 + dz^2 - (cdt)^2. \tag{1.11}$$

It may be remarked that this interval is invariant under Lorentz transformation. Thus, the geometry of the space-time continuum is characterized by Eqns. (1.11) interval (line element).

Remark: Since the time coordinate, here, is not just a fourth dimension but a unique one, we call the continuum $(3 + 1)$ dimensional space-time.
Signature of the line element:

If we observe the line element (1.11), the quadratic form ds^2 is characterized by the positive signs of dx^2, dy^2, dz^2 and a negative sign of dt^2. The sum of the positive and negative signs is called the signature of the line element, i.e., $+2$. If the line element is written as

$$ds^2 = -dx^2 - dy^2 - dz^2 + c^2dt^2 \tag{1.12}$$

then the signature is -2, and the signatures of the line element are usually represented by $(+,+,+,-)$ or $(-,-,-,+)$.

Remark: Here, we usually call the line element (1.12)
Space-like if $dx^2 + dy^2 + dz^2 > c^2dt^2$
Time-like if $dx^2 + dy^2 + dz^2 < c^2dt^2$
Singular (null) if $dx^2 + dy^2 + dz^2 = c^2dt^2$.

Thus the path of light rays from the event lies on the null cone. If the interval is space-like, we can always find proper coordinates in which the time component is zero and if the interval is time-like, space coordinates will be zero in proper coordinates.

Example 1-3 *If* (x_1, y_1, z_1, t_1) *and* (x_2, y_2, z_3, t) *are two events such that* $t_1 < t_2$ *does it follow that the first event precedes the second in an absolute sense?*

Sol.: From the Lorentz transformation Eqns. (1.2), we have

$$x_2' - x_1' = \beta[(x_2 - x_1) - v(t_2 - t_1)]$$
$$t_2' - t_1' = \beta\left[(t_2 - t_1) - \frac{v(x_2 - x_1)}{c^2}\right]. \qquad (1.13)$$

If the interval is space-like then $(x_2 - x_1) > ic(t_2 - t_1)$.

Now $x_2' > x_1'$ if $(x_2 - x_1) > v(t_2 - t_1)$ which immediately follows from the inequality since $v < c$.

$$t_2' > t_1'$$

if

$$t_2 - t_1 > \frac{v}{c^2}(x_2 - x_1) > \frac{v^2}{c^2}(t_2 - t_1)$$

since $t_2 > t_1$, and therefore it gives $c > v$ which is true.

Thus, for the Lorentz transformation when $t_2 < t_1$, t_2' is also $> t_1'$.

If we take $v = c$, then

$$x_2' - x_1' = \frac{x_2 - x_1 - c(t_2 - t_1)}{\sqrt{1 - \frac{v^2}{c^2}}} \approx \infty$$

since $(x_2 - x_1) > c(t_2 - t_1)$ and

$$t_2' - t_1' = \frac{(t_2 - t_1) - \frac{1}{c}(x_2 - x_1)}{\sqrt{1 - \frac{v^2}{c^2}}} \approx -\infty$$

Similarly, when $v = -c$,

$$x_2' - x_1' \approx \infty$$

but $t_2' - t_1' \approx +\infty$.

Thus, there is a coordinate system in which $(t_2' - t_1')$ can have any value from $-\infty$ to $+\infty$ - hence the temporal sequence of the events is not absolute.

Proper time

Suppose a particle is moving with a velocity V concerning an inertial system S and if S' and S are in uniform relative translatory motion, then the particle is at rest for S'. The time measured by a clock fixed in S' is called proper time. Also, if

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

is the interval between two events, then the proper time $d\tau$ is given by

$$d\tau = \frac{ds}{c} = \sqrt{1 - \{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2\}}dt$$

$$= \sqrt{1 - \frac{V^2}{c^2}} \qquad (1.14)$$

Remark: The concepts of absolute time interval dt and proper time interval $d\tau$ are related by the Eqn. (1.14).

World lines

We know that in Minkowski space, every event is denoted by a world point. The curve joining all the world points is called the world line and is usually written as

$$x^i = x^i(u) \qquad (1.15)$$

where u is a parameter of the curve and $i = 1,2,3, \ldots n$.

Light cone

It is well known that the interval between two events given by

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$$

is invariant under Lorentz transformation. If $dx^2 + dy^2 + dz^2 = 0$, it forms a surface which is called a light cone. A light signal which starts at the origin (0,0,0,0) of the system reaches every world point sooner or later, and hence the surface formed is a light cone. It divides Minkowski's space into two domains, with $ds^2 > 0$ the future and $ds^2 < 0$ the past.

1.4 Concepts of Relativistic Mechanics

Introduction: The postulates of the special theory of relativity and their consequences make it necessary to revise the concepts of classical mechanics. Newtonian mechanics developed on the idea of a body's motion depending on mass, velocity, acceleration, force, and time. Also in Newtonian mechanics, the basic principles of the conservation of mass and the conservation of momentum are valid with mass being constant. If the special theory of relativity and the conservation principles, namely

$$\sum m = const$$

$$and \sum m\vec{u} = const, \qquad (1.16)$$

are to be valid simultaneously, then our concepts of mass, momentum, energy, and force must be redefined, and the results are as follows:

Results in Relativistic Mechanics.

The mass of a moving particle: If S and S' are two systems which are in a relative translatory motion and if m is the mass of a body moving with a velocity u, then we should have

$$m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \tag{1.17}$$

where m_0 is the rest mass of the body or particle.

Remark: If $u = 0$, then $m = m_0$, which is called the rest mass or proper mass of the body. Also, if u is minimal compared to the velocity of light c, then $m = m_0$.

The principle of mass-energy equivalence ($E = mc^2$): If S and S' are in uniform, relative translatory motion and if a particle of mass m is also moving with velocity v, then we have the total energy as

$$E = T + m_0c^2 = mc^2 \tag{1.18}$$

where T is the kinetic energy of the moving particle and m_0c^2 is the rest energy of the particle or the internal energy.

Remarks: From Eqn. (1.18) we have the following results

i. If $v << c$, then we have

$$T = \frac{1}{2}m_0v^2 \tag{1.19}$$

which is the Newtonian limit of the result for Eqn. (1.18).

ii. $T = (m - m_0)c^2$ (1.20)

i.e., a change in inertial mass is equal to a change in kinetic energy.

iii. From Eqn. (1.18) we have

$$T = m_0c^2\left[\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right]. \tag{1.21}$$

Now, if $v \to c$, then $T \to \infty$, which means that an infinite amount of energy is needed to increase the velocity of a particle to that of the velocity of light.

Transformation equations for mass: Suppose S and S' are two systems which are in relative translatory uniform motion with velocity v and if m

is the mass of the moving body with velocities u and u_1 in S and S', then we have

$$m_1 = m\frac{\left(1-\frac{vu_x}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}} \tag{1.22}$$

where $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$. If $u_x = 0$, then

$$m_1 = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{1.23}$$

Transformation formulae for momentum and energy: If m and m_1 are the masses of a body in the inertial systems S and S', $u,$ and u' are the velocities, such that

$u = (u_x, u_y, u_z)$, $u' = (u_{x'}, u_{y'}, u_{z'})$

and if the momentums p_x, p_y, p_z are

$p_x = mu_x$, $p_y = mu_y$, $p_z = mu_z$

and E and E' are the energies in the systems S and S' respectively, then we have

$$p_{x'} = \beta\left(p_x - \frac{vE}{c^2}\right), \qquad p_{y'} = p_y, \qquad p_{z'} = p_z$$

and

$$E' = \beta(E - vp_x) \tag{1.24}$$

where $\beta = \left(1-\frac{v^2}{c^2}\right)^{\frac{-1}{2}}$.

Remark: From Eqns. (1.24) it is a simple matter to deduce the following results:

i. We can observe that in the Lorentz transformation equations given by (1.1), if we replace x, y, z, t by p_x, p_y, p_z, E/c^2, respectively, we obtain the transformation equations for momentum and energy given by Eqns. (1.24).
ii. It is also a simple matter to see, using (1.18) and (1.24), that $p^2 - \frac{E^2}{c^2}$ is Lorentz invariant, i.e.,

$$p'^2 - \frac{E'^2}{c^2} = p^2 - \frac{E^2}{c^2}, \tag{1.25}$$

which is equal to

$$p_x^2 + p_y^2 + p_z^2 - \frac{E^2}{c^2} = m^2(u_x^2 + u_y^2 + u_z^2) - \frac{m^2c^4}{c^2}$$
$$= m^2u^2 - m^2c^2$$
$$= \frac{m_0^2(u^2 - c^2)}{1 - \frac{u^2}{c^2}}$$
$$= -m_0^2c^2$$

$$\therefore E^2 = c^2(m_0^2 + p^2) \quad (1.26)$$

Example 1-4 *If a body of mass m disintegrates while at rest into two parts of rest masses m_1 and m_2, show that the energies E_1 and E_2 of the parts are given by $E_1 = \frac{c^2(m^2+m_1^2-m_2^2)}{2m}$, $E_2 = \frac{c^2(m^2-m_1^2+m_2^2)}{2m}$.*

Sol.: We know that

$$E_1 + E_2 = total\ energy = mc^2. \quad (1.27)$$

Just after disintegration, the masses m_1 and m_2 move off in opposite directions with equal momenta, i.e., if m_1 moves with momentum p, m_2 will move with momentum $(-p)$.

Hence from (1.26)

$$E_1^2 = c^2(m_1^2c^2 + p^2)$$
$$E_2^2 = c^2(m_2^2c^2 + p^2)$$
$$\therefore E_1^2 - E_2^2 = (m_1^2 - m_2^2)c^4 \quad (1.28)$$

Hence dividing (1.28) by (1.27), we obtain

$$E_1 - E_2 = \frac{m_1^2-m_2^2}{m}c^2. \quad (1.29)$$

Hence E_1 and E_2 can be obtained from (1.27) and (1.29).

Example 1-5 *A particle of mass M, at rest, decays into two smaller particles of masses m_1 and m_2. What are their energies and momenta?.*

Sol.: Before decay, four-momentum is $(E/c, p) = (Mc, 0)$. After the decay the two particles must have equal and opposite three-momenta p_1 and p_2 to conserve three-momentum. Define $p = |p_1| = |p_1|$; to conserve energy $E_1 + E_2 = E = Mc^2$ or

$$\sqrt{p^2 + m_1^2c^2} + \sqrt{p^2 + m_2^2c^2} = Mc$$

This quadratic equation can be solved numerically for p and then $E_1 =$

$\sqrt{m_1^2c^4 + p^2c^2}$ and $E_2 = \sqrt{m_2^2c^4 + p^2c^2}$.

Solve the above problem again for the case $m_2 = 0$. Solve the equations for p and E_1 and then take the limit $m_1 \to 0$. (This is left to the reader as an exercise).

Transformation equations for force: Let S and S' be uniformly moving systems with relative velocity along the x-direction, and let m and m' be the masses of a body referred to S and S' respectively which are moving with velocities u and u'. Also, let $\overleftarrow{F}$ be a force on a body of mass m and velocity u, then the transformation formulae for force components are

$$F_{x\prime} = F_x - \frac{v/c^2}{\left(1-\frac{vu_x}{c^2}\right)}(u_yF_y + u_zF_z),$$

$$F_{y\prime} = \frac{\sqrt{1-v/c^2}}{\left(1-\frac{vu_x}{c^2}\right)}F_y, \tag{1.30}$$

$$F_{z\prime} = \frac{\sqrt{1-v/c^2}}{\left(1-\frac{vu_x}{c^2}\right)}F_z.$$

The relativistic formula for density: If S and S' are two systems which are in a relative translatory uniform motion with velocity v, then we have

$$\rho' = \frac{\rho_0}{(1-\frac{v^2}{c^2})}. \tag{1.31}$$

where ρ and ρ' are densities in S and S' and ρ_0 is the rest energy density.

1.5 Four-dimensional language of relativistic mechanics

Introduction: With the advent of the special theory of relativity, we have seen that the concepts of Newtonian mechanics have been re-defined. We have also seen that we live in a four-dimensional continuum. Given the above, we write down four-dimensional expressions for force and equations of motion. For this purpose, we take

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2 \tag{1.32}$$

which describes the four-dimensional continuum with the Galilean coordinates

$$x^1 = x, x^2 = y, x^3 = z \ and \ x^4 = \mathrm{i}ct. \tag{1.33}$$

Now we define the following:

i. **Four-momentum:** It is given by

$$m_0 \frac{dx^\mu}{ds} = \left(m_0 \frac{dx^1}{ds}, m_0 \frac{dx^2}{ds}, m_0 \frac{dx^3}{ds}, m_0 \frac{dx^4}{ds}\right) \quad (1.34)$$

where m_0 is the rest mass and $\frac{dx^\mu}{ds}$ is the four-dimensional velocity vector. Then the four-momentum conservation is

$$\sum m_0 \frac{dx^\mu}{ds} = constant \quad (1.35)$$

where summation $\sum$ should be taken over all the particles of the system.

ii. **Four-Force: (Force four-vector)** The four-force vector is defined as

$$F^\mu = \left(\frac{F_x}{\sqrt{1-\frac{u^2}{c^2}}}, \frac{F_y}{\sqrt{1-\frac{u^2}{c^2}}}, \frac{F_z}{\sqrt{1-\frac{u^2}{c^2}}}, \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \frac{dE}{cdt} \right). \quad (1.36)$$

iii. Minkowski equation of motion is

$$F^\mu = c^2 \frac{d}{ds}\left(m_0 \frac{dx^\mu}{ds}\right) \quad (1.37)$$

where m_0 is the proper mass of the particle.

Exercise 1

1. Determine the speed with which one object must move relative to another for its clock to be slowed by 1%, as seen by the other.
2. You are driving at a steady 100 km/hr. At noon you pass a parked police car. At twenty minutes past noon, the police car passes you, traveling at 120 km/hr.
   a. How fast is the police car moving relative to you?
   b. When did the police car start driving, assuming that it accelerated from rest to 120 km/hr instantaneously?
   c. How far away from you was the police car when it started?
3. If you throw a soccer ball at speed v at a wall, it bounces back with the same speed, in the opposite direction. What happens if you throw it at speed v towards a wall traveling towards you at speed w? What is your answer in the limit in which w is much larger than v?
4. You are trying to swim directly East across a river flowing South. The river flows at $0.5\, msec^{-1}$ and you can swim, in still water, at $1\ msec^{-1}$. If you attempt to swim directly East, you will drift downstream relative to the bank of the river.
   a. What angle θ_a will your velocity vector relative to the bank make with the direction East?
   b. What will be your speed (magnitude of velocity) v_a relative to the bank?
   c. To swim directly towards the East relative to the bank, you need to head upstream. At what angle θ_c do you need to head, again taking East to be the zero of the angle?
   d. When you swim at this angle, what is your speed v_c relative to the bank?
5. A woman walks to a shop at a speed of $6\, km/hr$. How accurately would it be necessary to measure her walking stick to detect the Lorentz contraction of the stick if it is 1 m long? Is such an accuracy physically attainable?
6. How much slower (or faster) is the speed of light in the air relative to a vacuum? How do you think the speed will depend on temperature and pressure? How much slower (or faster) is the speed of light in glass and water relative to a vacuum.

7. How fast do you have to throw a meter stick to make it one-fourth of its length at rest?
8. Consider a clock, which, when at rest, produces a flash of light every second, moving away from you at $\frac{4}{5}$c.
   a. How frequently does it flash when it is moving at $\frac{4}{5}$c?
   b. By how much does the distance between you and the clock increase between flashes?
   c. How much longer does it take each flash to get to your eye than the previous one?
   d. What, therefore, is the interval between the flashes you see?
9. Two spaceships, each measuring $100\ m$ in their rest frame, pass by, traveling in opposite directions. Instruments onboard spaceship A determine that the front of spaceship B requires $5 \times 10^{-6} s$ to traverse the full length of A.
a. What is the relative velocity v of the two spaceships?
b. How much time elapses on a clock on spaceship B as it traverses the full length of A?
10. If light is affected by gravitation, it should be possible to go into orbit (i.e., a closed path) about a gravitational source. How dense would a thousand-kilogram mass have to be for light to be in a circular orbit about it?
11. An observer, A, sees a body as having twice the length that another observer, B, sees. Which of them has the greater speed relative to the body if it lies along the direction of their relative motion? If a third observer sees the length as three times that as seen by B and the body is in the rest-frame of one of the three observers, which is the rest-frame of the body?
12. Particles of half-life 10^{-8} secs are produced 3 km above sea level, and most of them are found at sea level. What is the least speed at which they must be traveling? In their rest-frame, there is no time dilation. How is it that they, nevertheless, can travel 3 km?
13. Determine the speed with which one object must move relative to another for its clock to be slowed by 2%, as seen by others.
14. Prove that the relativistic resultant of three co-linear speeds u, v, w is given by