Complicated Methods of Logical Analysis Based on Simple Mathematics

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... If there are controversies among people, it would be sufficient just to say “Let us calculate!” in order to ... make clear who was right.

Gottfried Wilhelm Leibniz

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FOREWORD

Our life is arranged so that we are constantly forced to convince each other of something, to encourage an interlocutor to take some action, to justify the expediency of our actions or inactions. This happens not only in everyday life, but also in many other spheres of human activity: science, production, politics, etc. Motivating means are different: sometimes these are force or threats; this has nothing to do with logic. They often convince an interlocutor with the help of deception, psychological influence or "neurolinguistic programming". This is also not logic. There is a wonderful book by Robert H. Thouless (1974) that tells in detail and fascinatingly about the tricks in a dispute aimed at confusing the interlocutor.

At the same time, in many cases, deception can be recognized by means of logical analysis. In everyday practice as well, every person needs logic, at least in order not to become a victim of verbal manipulations and to be able to critically analyse their delusions that cause us major or minor troubles.

Probably, it is impossible to find a person who would never make logical mistakes in her/his reasoning. Once upon a time, the basics of logic were taught in schools and gymnasiums, and the analysis of logical errors in the arguments of opponents played a significant role in science and education. However, the 20th century has noticeably faded the role of logic in human culture. Currently, logic specialists are mainly engaged in the invention or research of numerous "exotic" logics. As a result, the essence of logic has ceased to be clear to many.

There is still a lot of people interested in knowing what logic is and what it is for. There are many answers to this question. We prefer this one: ***logic is the most important component of general human culture; its main purpose is to develop correct methods for analysing the correctness of reasoning and justifications***.

Correctness of methods of logical analysis is currently interpreted ambiguously. Many believe that these methods are correct due to the fact that they have been tested by centuries of practice in applications of logic. For the logic, which is the foundation of human cognitive abilities, such an "empirical" criterion is clearly insufficient. We will use a different point of view: ***logical methods are correct to the same extent, to which they are mathematically grounded***.

There are still discussions among logic specialists about how to teach logic. Basically, these discussions revolve around the problem of the relationship between logic and mathematics. Nowadays, many mathematicians believe that logic and mathematics as a whole are based on an artificial language incorporating some symbols to denote variables, constants, functions, predicates, logical connectives, brackets, and punctuation marks. For these symbols, methods are formulated to construct correct statements and to transform them into other correct sentences. This approach gained popularity among mathematicians at the turn of the 19^{th} and 20^{th} centuries, it is called *theory of formal systems* (TFS). Another name is also used, it is the *axiomatic method*. Examples of some important sections of logic and mathematics presented under the influence of this approach can be found in publications of Bourbaki (2004) and Mendelson (1997).

From the point of teaching logic, such approach has a number of disadvantages: it is difficult for students to assimilate and is poorly suited for analysis of natural reasoning and solving logical problems. In addition, from the point of view of TFS, classical and non-classical logics have equal rights to exist, and the question of which logic is "correct" is practically not discussed among specialists.

Our proposed methodology, the main provisions of which are published in scientific publications (Kulik 2001, Kulik and Fridman 2017), sets out a different approach, where main logical relationships and methods of logical analysis are based on simple mathematical structures. This makes it possible to use plain methods similar to calculations for analysis of reasoning and, in addition, allows students to master some of the fundamental concepts of modern mathematics, which are currently used not only in logic, but also in many other areas including information technologies.

The first part of the book examines methods for analysing reasoning within the framework of syllogistics created in the 4^{th} century BC by the ancient Greek philosopher Aristotle. An impetus for development of the methods proposed here to analyse syllogisms (reasoning with two premises) and polysyllogisms (reasoning with an arbitrary number of premises) was the acquaintance of the authors with the wonderful Lewis Carroll's[1] book "Symbolic Logic" (1958) that contains a big section devoted to analysis of natural reasoning. Despite the fact that mathematical methods of analysis used then by Carroll are outdated nowadays, and more modern methods are proposed here, some of the L. Carroll's undeservedly forgotten ideas formed the basis of our new approach.

It is important to note that polysyllogistics does not cover all the methods currently used to analyse reasoning. Other, sometimes much more complex, methods of logical analysis were strictly justified within the framework of mathematical logic, which began to develop rapidly from the middle of the 19th century. However, reasoning in the form of syllogisms is very often used in everyday practice. In addition, a detailed examination of polysyllogistics is not only a tribute to the system of reasoning that has existed for more than two millennia, but also an opportunity to trace some close connections between mathematics and logic.

Methods for analysing reasoning that are more capacious in their analytical possibilities are presented in the second part "*N*-tuple Algebra and Logic". These methods are often used in both mathematical logic and artificial intelligence systems.

The book presents new approaches to logical analysis developed by the authors, based on the well-known mathematical system – algebra of sets. For comparison, Appendix C provides an overview of traditional logical systems: Aristotle's syllogistics, propositional calculus, and predicate calculus theory. Some information on non-classical logics is also given there.

The methods to analyse reasoning proposed in the first part of this book, "Polysyllogistics", have the following advantages relative to the traditional syllogistics:

1) they do not contain errors of traditional syllogistics;

2) with their help, arbitrary sets of propositions are easily analysed;

3) they allow to analyse logical incorrectness and to test hypotheses;

4) they make it possible to restore the missed premises, i.e., to provide abductive inference.

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[1] Lewis Carroll is a literary pseudonym for the English mathematician and logician C.L. Dodgson.

PREFACE

The main topic of this book is to demonstrate capabilities of two relatively simple mathematical systems, *E*-structures and *n*-tuple algebra, in solving logical problems of reasoning and justification. Descriptions of these structures and this algebra are presented in numerous scientific publications of the authors. Many included illustrations make the text of the book more accessible to the reader.

In the first part of the book, properties of Aristotle's proposition in syllogistics are compared with properties of the inclusion relation in algebra of sets. In the following sections, the basics of set algebra and some types of binary relations (graphs and partially ordered sets) are presented in a simple and clear language. The authors proposed an interesting "combinatorial method", in which, instead of the basic definitions and axioms adopted in the traditional presentation of mathematical structures, a substantiation of the laws of set algebra is accomplished only by using the definitions of its operations and relations. The proposed approach is intuitive and appropriate in popular science literature, to which this book belongs. The same can be said about the prohibition of using sets as elements proposed in the book, which, in the authors' opinion, allows to expel some paradoxes of set theory.

After describing the considered foundations of algebra of sets, *E*-structures are defined on the basis of properties of the inclusion relation of set algebra. In particular, they use the laws of transitivity, contraposition and double complement as inference rules. These laws allow deducing all possible consequences for universal statements like "All *A* are (not) *B*". To obtain conclusions with particular statements of the type "Some *A* are (not) *B*", one of the properties of the upper cones in partially ordered sets is used, which in terms of algebra of sets can be formulated as follows: "two sets *A* and *B* have a nonempty intersection if they are included in the upper cone of a non-empty set".

To simplify analysis of polysyllogisms, the authors proposed a graphical method that allows to simply and visually display not only the process of logical inference, but also the analysis of possible inconsistencies in reasoning, which are called collisions. Capabilities of this method are shown in formation and analysis of hypotheses, as well as in generating and checking abductive conclusions.

The final section of the Part I introduces a logical analysis of metaphors based on the proposed methodology. Here, the paradox of masquerade is considered, which is formally close to the metaphor and may appear in reasoning by analogy.

In the second part of this book, a more complex field of application of logical analysis is considered, for which methods of mathematical logic are currently used. The methodology proposed by the authors is also based on the laws of set algebra, but here they are applied for *n*-ary relations. In this technique, an important role belongs to the properties of the Cartesian product of sets. The authors succeeded, due to a well-chosen designation of this mathematical object and identification of new connections between the Cartesian product and the formulas of mathematical logic, in constructing a system called *n*-tuple algebra (NTA). The properties of this system, in particular, algorithms for operations of algebra of sets and for checking equalities or inclusions for NTA-objects are rigorously proved.

Section 3 of Part II shows capabilities of NTA in solving some logic problems. It is worth to note here the possibility of relatively easy expressing conditions of problems with uncertainties in the form of variants of values in the language of NTA.

Section 4 shows how to extend the scope of NTA by introducing some simple operations with attributes; in particular, it becomes possible to comparatively easily perform various operations with relations that have different relation diagrams.

The fifth section displays how to express some basic formulas of propositional and predicate calculi by using NTA. The NTA capabilities in performing logical analysis are shown for not only deductive methods, but also for defeasible reasoning. Here the interest is aroused by developing a unified approach for solution of such problems. Also of interest is the method proposed by the authors for derivation consequences with predetermined properties.

The last section of Part II provides some estimates of computational complexity of the main NTA algorithms.

The book is written in a relatively simple and clear language, especially its first part. As for the second part, the NTA can scarcely be called an easily assimilated mathematical system. Nevertheless, the advantages of this system include a strict and lucid sequence of presentation as well as fairly grounded and transparent initial premises.

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List of Reference Designations and Acronyms

$\circ$ – composition, one of the NTA generalized operations
$\triangleright\triangleleft$ – join, one of NTA generalized operations
$\varnothing$ – an empty set or an empty component of an NTA object
$*$ – complete component of an NTA object
$\subseteq_G$ – inclusion, one of NTA generalized relations
$\cap_G$ – intersection, one of NTA generalized operations
$\subset_G$ – strict inclusion, one of NTA generalized relations
$\cup_G$ – union, one of NTA generalized operations
+*Attr* – addition a dummy attribute, one of NTA operations on attributes
$=_G$ – equality, one of NTA generalized relations
AI – artificial intelligence
-*Attr* – elimination of an attribute, one of NTA operations on attributes
CNF – conjunctive normal form
CP – Cartesian product of sets
CSP – Constraint Satisfaction Problem
DNF – disjunctive normal form
E-structure – Euler's logical structure
Gen – the rule of generalisation
KB – knowledge base
M.P. – member of Parliament
MPP – massive parallel processing
NP – non-deterministically polynomial
NP-hard problems – problems more complex than *NP*-complete ones, i.e., enumeration problems
NTA – *n*-tuple algebra
poset – partially ordered set
QC-structure (from *Q*uasi-*C*omplement) – a poset with quasi-complements
SAT – satisfiability problem for a given CNF
SMP – symmetric multiprocessing architecture

TFS – theory of formal systems
U – the universe

Part I.

Polysyllogistics

1. Proposition

At the heart of logic, there is a structure called “proposition” or “statement”. This concept was developed by Aristotle (384-322 BC). He created a system to analyse reasoning, now known as *syllogistics*.

In general case, a ***statement*** is a certain form of expression that formulates a fragment of our knowledge, ideas or opinions about the surrounding world (for instance, “Some mushrooms are poisonous”, “All metals are electrically conductive”). In Aristotelian syllogistics, each statement consists of two parts called “subject” and “predicate”. The logical meaning of a statement lies in the fact that the “predicate” is considered as an inherent feature or condition of existence for a given “subject”. The “predicate” can also be expressed in negative form (for example, “The penguin *does not fly*”). The “subject” of a proposition is usually accompanied by logical “prefixes” (in logic, they are called ***quantifiers***) “All” or “Some”. In Aristotle's syllogistics, as a rule, negation is applied to predicates rather than subjects, and quantifiers are applied only to subjects. Subjects can be individual (Sherlock Holmes, Liverpool, etc.) or general (literary heroes, cities, etc.).

In our approach, we use statements in a broader sense. First, a proposition may include more than one predicate and, second, negations may be used for both predicates and subjects. By the way, L. Carroll also uses these assumptions in his “Symbolic logic” (1958).

Traditional syllogistics admits only four forms of statements, namely “All *A* are *B*”, “All *A* are not *B*”, “Some *A* are *B*”, and “Some *A* are not *B*”. These forms (or types) were singled out by Aristotle. They correspond to the usual sentences expressing relationships between part and whole, species and genus, object and property, etc. If you look closely at the sentences of a natural language, many of them occur to be presentable in the form of such statements without loss of meaning. We can say that the grammatical and semantic structure of the sentence, formed over many millennia of human existence, was embodied in the logical statement.

A significant part of sentences in our speech consists of a subject and a predicate. They are often accompanied by minor members of the sentence (definitions, additions, circumstances of place, time, etc.). In this generally accepted grammatical form, many sentences are expressed in everyday conversations and literary works, as well as in philosophical and scientific reasoning in all national languages. In linguistics, there are many approaches to analysis of the structure of a sentence. Here we will consider a variant in which three main structural elements of a simple sentence are distinguished: the subject, the definition and the predicate, together with additions and circumstances that are under its control (for example, “was born in London”, “was out of work”, “will arrive on Tuesday”, “is an IBM representative”, etc.).

When analysing the logical form of a statement, the subject can often be considered as the subject of the statement. Statement predicates are constructs consisting of predicates with circumstances or additions controlled by them. For example, in the sentence “Kangaroos live in Australia”, the subject is the kangaroo as one of the animal species, and the predicate is the creatures living in Australia.

In this aspect, it is more difficult to resolve the issue with definitions expressed in a language by adjectives, subordinate clauses or participials. Usually, they limit the volume of the subject, that is, they highlight only a part of the object designated by the subject, for example, “black swans”. In this case, it is advisable to choose the subject as a term with a definition. Conversely, the definition itself can be used as an additional predicate. For instance, the sentence “Black swans swim in a pond” can be expressed as the propositional form in two ways: 1) the subject is “black swans”, and the predicate is “creatures swimming in a pond”; 2) the subject is “swans”, and predicates are “black objects” and “creatures swimming in a pond”.

Note that the form of a statement allows to express not only many sentences of a natural language, but also such logical constructions as definitions or interpretations of terms; real life facts; many mathematical theorems; laws of nature, etc. Each predicate of a statement is a necessary feature or a necessary condition for existence of its subject.

Analysis of numerous examples of statements shows that predicates can be represented as some sets of objects with certain properties, and subjects are either sets of objects with certain properties or individual objects. Therefore, relations between subjects and predicates in statements, as well as methods to analyse reasoning, can be easily modelled within a mathematical system called ***set algebra***.

It should be stated that attempts to use set algebra (algebra of sets) for analysis of syllogisms and polysyllogisms have been made earlier several times (works by J.D. Gergonne, J. Venn, and others). However, they did not show significant advantages when compared with traditional syllogistics. Some progress in this issue was achieved through the use of only three laws of set algebra (transitivity, contraposition and involution), as well as some elementary properties of partially ordered sets. This is discussed in more detail in the following sections.

2. Basic Concepts of Algebra of Sets

Algebra of sets underlies many branches of modern mathematics. It is almost impossible to establish an exact date of its inventing and to name the discoverer. It has been gradually developing against the background of numerous attempts to find a rigorous mathematical foundation for Aristotelian logic. Some prerequisites for this algebra are contained in the works of Leibniz, many famous logicians and mathematicians: L. Euler, J.D. Gergonne, A. de Morgan, J. Venn, etc. (Stiazhkin 1967) significantly contributed to this scientific field.

Among the branches of mathematics, algebra of sets turns out to be much simpler than, say, geometry, differential and integral calculi, but in its complete form it was expressed only in the second half of the 19th century, that is, much later than many, sometimes very complex, fields of mathematics.

The centuries-old history of interactions between logic and mathematics is described in detail in the book of Stiazhkin (1967). One of the key events in this history was the Euler's book (1840), in which he expounded his understanding of Aristotle's syllogistics in a popular form. In this book, he used visual diagrams, which were later called “Euler diagrams”. Later, Euler's diagrams were used not only in training courses on logic, but also in outlining the basics of many fundamental branches of modern mathematics where set algebra was applied (for example, in Kolmogorov and Fomin (1957)). Here we will also use these visual mappings that allow to quickly master the abstract concepts of set algebra.

Euler's ideas were developed in the works of the French astronomer and mathematician J.D. Gergonne. In his work "Foundations of Rational Dialectics" published in 1817, he succeeded in presenting all the classes of statements identified by Aristotle with the help of relations between sets. These correlations are called "Gergonne's relations" (Stiazhkin 1967). Consider them in more detail.

Syllogistics exploits simple statements of four types:

A is the universal affirmative proposition (all *X* are *Y*);

E is the universal negative proposition (all *X* are not *Y*);

I is the particular affirmative proposition (some *X* are *Y*);

O is the particular negative proposition (some *X* are not *Y*).

The terms *X* and *Y* can be displayed as some aggregates (sets, classes) in the form of Euler diagrams. Gergonne identified five possible relationships between them (Fig. 1).

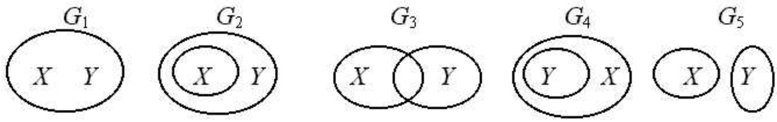

Figure 1.

Each type of Gergonne relations has its own name:

G_1 means coincidence or equivalence;

G_2 denotes left-hand switching;

G_3 stands for partial matching;

G4 names right-hand inclusion;

G5 displays incompatibility.

Gergonne showed that each type of Aristotelian propositions corresponds to some types of these relations, in particular:

- type **A** corresponds to G_1 or G_2;
- type **E** equals G_5;
- type **I** is equivalent to G_1 or G_2 or G_3 or G_4;
- type **O** is equal to G_3 or G_4 or G_5.

For instance, a statement of type **I** means that some non-empty part of a set or class *X* is contained in *Y*. Looking at the Fig. 1, it is easy to see that this condition is satisfied by all types of Gergonne relations, except for G_5. In logic, the word "some" is used in a broader sense: "at least one, but it is possible that all". The exact synonym for "some" is the term "exists".

Gergonne's relations were often used to rigorously substantiate not only the rules of inference from simple syllogisms, where only two propositions are used as premises, but also for more complex inferences, when premises can include an arbitrary number of statements; in this case we are dealing with a polysyllogism. The pinnacle of this kind of analysis can be considered the work of the English logician and philosopher J. Venn (1834–1923) (Stiazhkin 1967).

However, it is not always easy to analyse reasoning by using Gergonne's relations. The main difficulty is that for almost all types of statements (with the exception of the type **E**), several variants of these relations must be used, and with an increase in the number of initial statements, the number of possible variants for analysis increases like an avalanche. If we, for example, analyse a complex reasoning containing many statements, then for each statement it is necessary to look through all the options of Gergonne's relations corresponding to it. Below, we will study much simpler ways to analyse reasoning, which are also based on the laws of set algebra and on some properties of partially ordered sets.

From the point of view of modern mathematics, ***algebra of sets*** belongs to the class of ***algebraic systems***, that is, the structures, which incorporate the following constituents:

1) a ***carrier*** is a set of objects (for instance, numbers, symbols, geometric shapes, sets, etc.);

2) a set of ***relations*** (for example: more, less, equal, etc.);

3) a set of ***operations*** (as addition, multiplication, intersection, and so on);

4) the basic ***laws***, which connect relations and operations (e.g., the invariability of the result of the multiplication of two numbers when the factors are swapped).

In algebra of sets, the ***carrier*** is a certain collection of sets. The basic concepts of set algebra are a ***set*** and an ***element***. The relationship between them is called the ***membership relation*** and is denoted by the sign "$\in$". The notation $b \in A$ is translated from the symbolic language as “b is an element of the set A” or “the element b belongs to the set A”. If all elements of a set can be enumerated (for example, it consists only of the elements a, b, and c), then the following notation is generally accepted for the set:

$A = \{a, b, c\}$.

The listed elements of a set are usually enclosed in *braces*.

Sets can be specified in two ways: by *formulating characteristic features* (for instance, the set K contains only non-negative even numbers not exceeding 8) or by enumerating elements (the mentioned set $K = \{0, 2, 4, 6, 8\}$).

Modern mathematics does not provide any clear definition of the membership relation yet. It is believed that an element can be any object, including a set. This assumption serves as a source of paradoxes; these paradoxes were discovered at the turn of the 19th and 20th centuries, and since then discussions on this topic have not subsided. Here we will adhere to the point of view, in which the membership relation links two different types of objects (“element” $\in$ “set”), but in no case should there be a relationship of the type “set” $\in$ “set”. Then uncertainties and paradoxes do not arise.

Between the sets is established another, at first glance, similar, but actually fundamentally different, relation. It is the ***inclusion***, structural properties of which in modern mathematics are defined quite clearly and unambiguously. Let us consider it in more detail. Two different variations of this relation are allowed:

“$\subset$” stands for “strictly included”;

“$\subseteq$” denotes “included or equal”.

The notation $A \subset B$ means that the set A is included in the set B, that is, all the elements of the set A are simultaneously elements of the set B, but the equality of these sets is impossible. For example, “fish” are included in “animals”, but are not equal to them. The notation $A \subseteq B$ says that the set A is included in the set B, and these sets may be equal too. The picture of the strict inclusion relation ($A \subset B$) with using Euler circles is shown in Fig. 2.

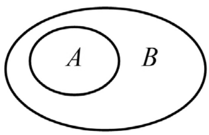

Figure 2.

In this case, it is not necessary to use the correct circles. Sets can be drawn with any closed shape.

If the sets are given by enumerating elements, the relation of inclusion (or non-inclusion) of one set into another set can be easily established for sets with a small number of elements by comparing these elements. For instance, if the following three sets are given:

$P = \{a, b, c, d, e\}$; $Q = \{b, d, a\}$; $R = \{a, c, f\}$,

by comparing their elements, we can prove that $Q \subset P$, while the relation $R \subset P$ is false, since the element f from the set R does not belong to the set P. For sets with a large or infinite number of elements, the inclusion relation can sometimes be justified. E.g., it is relatively easy to prove that the infinite set of all numbers divisible by 6 is strictly included in the infinite set of all even numbers.

The order of enumerating the elements of sets is irrelevant. For instance, the sets $\{b, d, a\}$; $\{a, b, d\}$; $\{d, a, b\}$ are essentially the same set. If the order of enumeration of some sets does matter, we deal with *sequences* or ***ordered sets*** (some information about them is given below) rather than sets. Mathematical properties of sequences significantly differ from those of sets.

The difference between the membership relation and inclusion can be illustrated by the following example. Suppose there is the notation $a \in P$. It yields that a is an element, and P is a set. Is it possible to write $a \subseteq P$ in this case? The latter record turns out to be wrong since the inclusion relation is applicable for two sets only. For such a case, the correct notation is $\{a\} \subseteq P$ where the left part is not an element, but a *one-element set*.

Consider another relation between sets, namely the ***equality relation***. Sets with the same elements are equal. Proving the equality of two sets, especially ones with large or infinite number of elements, is done by using the next statement:

The set A ***is equal*** to the set B, if both $A \subseteq B$ and $B \subseteq A$.

Note that in mathematics this statement is a proven theorem. A set A is a ***subset*** of another set B, if $A \subset B$ or $A \subseteq B$. The set A itself is also considered its subset. In other words: for any set A, $A \subseteq A$.

In algebra of sets, another set is also especially distinguished and often used. It is called the "***empty set***" and denoted by the symbol "$\varnothing$". Intuitively, the empty set means a set that does not contain any elements. However, this intuitive definition does not fully reveal its essence and role in algebra of sets. Properties of the empty set will become clearer when we consider the operations of set algebra.

If a set is given by an enumeration of its elements, we can write down the *collection of all subsets* of this set. For instance, such a collection comprises eight subsets for the set $A = \{a, b, c\}$:

$\varnothing$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{a, b, c\}$.

The following simple relation was proved to immediately find out the total number of all subsets for a set with N elements. For any N, this number is equal to 2^N. For example, the above set A has 2^3 subsets. In mathematics, a given collection of sets is called a ***system of sets***.

A meticulous reader is already ready to expose incorrectness of the authors: earlier they argued that an element cannot be a set, but is not a system of sets equivalent to a set of sets? Well, let "collection" be said instead of "set", but what is the difference, in essence?

To avoid inconsistency, it is proposed to define a system of sets as a set of names (in the general case, designations) of subsets of a set. For instance, the notation $\{a, c\}$ denotes a set. And what is unusual in the fact that this designation reveals the content of the set? The name "airplane" also partially reveals the content of this object.

In addition to the empty set, any system of sets also includes the ***universe***, that is, the set containing all the sets of the set system as subsets. In other words, a system of sets is a given set of designations for subsets of a set taken as a universe. For example, sets of planets, comets, stars, etc. can have "astronomical objects" as a universe.

Sometimes, different universes are admissible in reasoning for the same set of objects. E.g., either the set of predators or the set of mammals can be chosen as the universe for lions and jackals. In order to avoid mistakes in such cases, it is better to choose a suitable universe in advance and not change it in the process of reasoning.

There is no standard designation for the universe. In what follows, we will denote it by the symbol $\boldsymbol{U}$.

Let us move on to operations. We start with the complement operation, which can be determined only when a universe is given for a set system.

Definition 1. The ***complement*** of a set A is a set of all elements that belong to the universe and do not belong to the set A.

The complement of a set logically corresponds to the negation "NOT". For instance, "not black" denotes any colour except for the black one. Usual notation for the complement of a set is a bar above the name of the set. E.g., $\overline{P}$ is the complement of the set *P*.

Example 1. $\boldsymbol{U} = \{a, b, c, d\}$ and $P = \{a, c\}$. Then $\overline{P} = \{b, d\}$.

Let us define two more basic operations, namely intersection and union of sets.

Definition 2. The set *C* is the ***intersection*** of two sets *A* and *B*, if it includes all elements of these both sets.

Usually, notation for an intersection of two sets is the symbol "$\cap$" between names of these sets. Thus, $C = A \cap B$.

So, the intersection of the set of all students of a given university and the set of all participants in Olympiads equals to the set of students from this university who took part in Olympiads. The same way, after intersecting the set of integers divisible by 2 with the set of integers divisible by 3 we obtain the set of integers divisible by 6.

Logically, this operation is equivalent to the connective "AND" (its notation is "$\wedge$" or "&"). If we consider objects with some properties *P* or *Q* and denote these objects as S_P and S_Q correspondingly, the logical formula $P \wedge Q$ will select only objects that have both of these properties, they will belong to the set $S_P \cap S_Q$.

Example 2. If $A = \{a, b, c, d\}$ and $P = \{a, c, f\}$, $A \cap P = \{a, c\}$.

Calculating of the intersection for two sets may show that these sets have no common elements. So, their intersection is the empty set. For instance, the intersection of two sets $Q = \{a, c,\}$ and $R = \{b, d\}$ equals to $\varnothing$.

In particular, this example illustrates one "unusual" property of the empty set: *the empty set is included in any set*.

This property is easy to prove. Indeed, let an arbitrary set *A* be given. For it and its given universe, there exists its complement $\overline{A}$, and Definitions 1 and 2 yield $A \cap \overline{A} = \varnothing$. Since the intersection of two sets is included in each of these sets, we obtain $\varnothing \subseteq A$.

Definition 3. The set *C* is the ***union*** of two sets *A* and *B*, if it contains all elements that belong to at least one of these sets.

Notation for the union operation for sets is usually the symbol "$\cup$", and Definition 3 can be written as $C = A \cup B$.

This operation corresponds to the logical connective "OR" designated as "$\vee$". For some sets of objects, which have properties *P* or *Q*, the logical formula $P \vee Q$ selects all objects with at least one of these properties. The objects with both properties P and Q belong to the union as well.

Example 3. For two sets $A = \{a, b, c, d\}$ and $P = \{a, c, f\}$, $A \cup P = \{a, b, c, d, f\}$.

Note that in Example 3, the elements a and c, which are contained in each of the sets A and B, are not doubled in the union C, but are included only once because all elements of any set must be different by definition. In mathematics and its applications, sets with multiple elements are sometimes used (they are called *multisets*). In such sets, some of the laws differ from the laws of ordinary set algebra.

The above-introduced operations of complement, intersection, and union constitute the ***basic operations*** of set algebra. By using them, several other (***derivative***) operations are defined. Consider one of the derivative operations, namely the difference operation.

Definition 4. The set C is the ***difference*** of sets A and B (denoted as $C = A \setminus B$,), if it includes only those elements of the set A that do not belong simultaneously to the set B.

Example 4. For $A = \{a, b, c, d\}$ and $B = \{a, c, f\}$, $A \setminus B = \{b, d\}$.

As we have mentioned, the difference operation can be expressed via basic operations: $A \setminus B = A \cap \overline{B}$.

If we define a universe, for example, $\boldsymbol{U} = \{a, b, c, d, e, f\}$, in Example 4, we can check this equality as follows:

$\overline{B} = \{b, d, e\}$; then $A \setminus B = A \cap \overline{B} = \{b, d\}$.

At the same time, the complement operation can be expressed by using the difference operation: $\overline{A} = \boldsymbol{U} \setminus A$. In Fig. 3, the corresponding operations on the sets are depicted by using Euler's diagrams. The results of these operations are shown in grey colour.

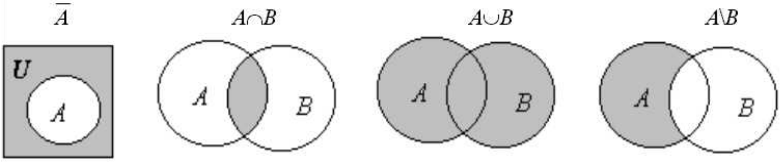

Figure 3.

It is worth to note here one important circumstance. For two sets A and B with no common elements, the following relations hold:

$A \cap B = \varnothing$; $A \subseteq \overline{B}$; $B \subseteq \overline{A}$.

These relations are displayed by Euler diagrams in Fig. 4.

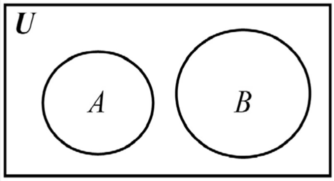

Figure 4.

Now we have enough concepts to formulate propositions mathematically. For example, we can divide the proposition “All members of the House of Lords hold the title of peerage” into the subject “members of the House of Lords” (*A*) and the predicate “hold the title of peerage” (*B*). Then this statement turns into the formula:

$A \subseteq B.$

It means that all members of the House of Lords are included in the set of those who bear the peerage. A more complex proposition, such as: “All Lords hold the title of peerage and are sane”, is expressed with using two predicates: “hold the title of peerage” (*B*) and “are sane” (C). We get the following mathematical formulation:

$$A \subseteq (B \cap C). \quad (1)$$

In cases where a statement contains predicates with negatives, the complement operation is used in the mathematical notation. For example, the statement “All Lords hold the title of peerage and do not take part in donkey races” can be written as

$$A \subseteq (B \cap \overline{D}), \quad (2)$$

where D is the predicate “take part in donkey races”.

If we use Euler diagrams, the visual representation of formulas (1) and (2) will look like in Figures 5 and 6.

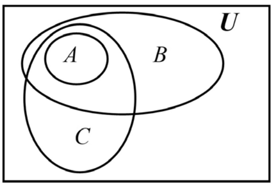

Figure 5.

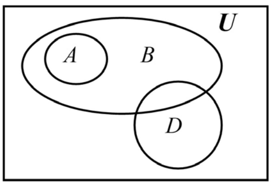

Figure 6.

Quantitative ratios (i.e., in this case, areas of figures) are not taken into account in Euler diagrams. In our knowledge, there are many such fragments where we do not know the number of elements of a set, but this does not prevent us from understanding that some of such sets are strictly included in other sets (for instance, a set of trees is included in a set of plants), or some sets do not have common elements with other sets (for example, there are no mammals among fish).

Many sentences of natural language can be translated into the mathematical form of statements. We will consider the mathematical and logical properties of statements in detail some later. In the meantime, we will give the general laws of set algebra, which are necessary for a deeper understanding of these properties.

3. Laws of Set Algebra and Their Justification

Laws of set algebra are essentially theorems derived from basic definitions and axioms. Often, 26 or 28 such laws are listed. Here, we present only a few of them, which are necessary for understanding what follows, and do not provide their proofs on the same reason. Eventually, we will show how these laws can be substantiated.

Let A, B, and C be some arbitrary sets in the universe $\boldsymbol{U}$. Then the laws of algebra of sets are the following relations between them.

1. $\overline{\overline{A}} = A$ is the double complement law.

Example 5. Let $\boldsymbol{U} = \{a, b, c, d\}$ and $P = \{a, c\}$. Then $\overline{P} = \{b, d\}$ and $\overline{\overline{P}} = \{a, c\} = P$.

In set algebra, this relation is known as the ***law of involution***. In logic, it is named ***negation of the negation*** (or the law of double negation): not(not-A) coincides with A.

2. $A \cap \overline{A} = \varnothing$ (the same elements may not be in both a set and its complement).

In logic, this law is called the ***law of non-contradiction*** corresponds to the law of consistency (any statement and its exact negation are logically inconsistent).

3. $A \cup \overline{A} = \boldsymbol{U}$.

In logic, this law corresponds to the ***law of the excluded third*** (or excluded middle, i.e., there may not be any third intermediate option above a statement and its complete negation.