Pure and Applied Algebraic Topology

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Chapter 1

Algebraic Topology

1 History

For many people, Poincaré's works [59] are the real foundation of the algebraic topology, when he defined for the first time in 1985 what is meant by homologous chains in a manifold. His definition was rather imprecise, but the notion he used covered exactly the current acceptance: two closed chains are homologous if they differ by an edge.

By 1940, the homological algebra theory was well defined and contributed greatly to the emergence of many other concepts like categories and functors. Various generalisations have been imagined later, like cohomology of groups with many surprising geometrical connections, bounded cohomology and equivariant cohomology. This shows, how homological notions have become largely widespread in almost all mathematics areas, and sometimes even in theoretical physics. The main principle of algebraic topology is to associate, in a functorial way, to any topological object an algebraic object which is invariant under certain kinds of transformations like homeomorphisms, homotopisms, holomorphisms and isomorphisms.

A constructive example is how to apply to a torus, two scissors to make it homeomorphic to a paper sheet. Topologically speaking, scissors here symbolise loops. Two cuts are said to be equivalent, when they have the same effect on the ambient space. In other words one can continuously switch from one to the other. The first cut goes around the central hole; and we get a crown. The second one is a radial cut on this crown and gives us a rectangle.

2 Functors and categories

Definition 1.1

A *category* is a collection of objects connected by a kind of arrows, called *morphisms* such that

1. $(f \circ g) \circ h = f \circ (g \circ h)$, for any morphisms f, g, h, whenever the composition is possible;
2. For any object X, there exists an unique morphism, denoted $id_X \in \hom(X, X)$ such that $id_X \circ f = f \circ id_X = f$, for any other morphism f, whenever the composition is possible.

As example of categories, one may consider sets connected by maps, topological spaces connected by continuous maps, groups connected by morphisms of groups, or finally vector spaces connected by linear maps.

Definition 1.2

A *functor* T between two given categories $\mathcal{C}$ and $\mathcal{C}'$ is any correspondence

$$T\colon \mathcal{C} \longrightarrow \mathcal{C}',$$

that associates to any object X in $\mathcal{C}$, an object $T(X)$ in $\mathcal{C}'$, and associates to any morphism $f\colon X \longrightarrow Y$ in $\mathcal{C}$, a morphism $T(f)\colon T(X) \longrightarrow T(Y)$ in $\mathcal{C}'$, such that

$$T(id_X) = id_{T(X)},$$

- for any X in $\mathcal{C}$.

Vocabulary. Let $T\colon \mathcal{C} \longrightarrow \mathcal{C}'$ be a functor.

- T is said to be covariant when $T(f \circ g) = T(f) \circ T(g)$, for any f, g;
- T is said to be contravariant when $T(f \circ g) = T(g) \circ T(f)$, for any f, g;

Examples.

1. Let $\mathcal{C} = Gpe$, be the category of groups endowed with morphisms of groups, then $T(g) = g^{-1}$ is a contravariant functor.
2. Let $\mathcal{C} = Gpe$, be the category of groups endowed with morphisms of groups, then $T(g) = h^{-1}.g.h$ is a covariant functor, where h is a fixed morphism.

3 Homology

Definition 1.3

A *chain complex* is any $\mathbb{N}$-indexed family $(C_n)_{n\in\mathbb{N}}$ of modules endowed with a family of morphisms of modules

$$d_n\colon\ C_n \longrightarrow C_{n-1},$$

such that

$$d_n \circ d_{n+1} = 0,$$

with the convention that $C_{-1} = 0$.

Following the notation here above, $Z_n\colon = \mathrm{Im} d_{n+1} \subset B_n\colon = \ker d_n$. Elements of Z_n are called *n-cycles*, while those of B_n are called *n-bords*. From $Z_{n+1}\colon = \mathrm{Im} d_{n+1} \subset B_n\colon = \ker d_n$, we deduce that any cycle is a bord. The inverse is naturally not always true.

By setting $C\colon = \bigoplus_{n\in\mathbb{N}} C_n$, we get a graduation: any element $c \in C_n$ is called of degree n and we write $|c| = n$. This yields to the map

$d\colon C \longrightarrow C$ where $d_{|C_n} = d_n$. In particular $d^2 = 0$, which means that d is a derivation, called a *differential*.

Definition 1.4

Let $(C,d) := \bigoplus_{n\in\mathbb{N}} C_n$ a chain complex.

- $H_n(C,d) := \ker d_n / \mathrm{Im} d_{n+1}$ is called the n-th group of homology of (C,d);
- $\beta_n(C) := \mathrm{rank H_n(C,d)}$ is called the Betti number of (C,d);
- $H_*(C,d) := \bigoplus_{n\in\mathbb{N}} H_n(C,d)$ is the homology group of (C,d);
- $\dim H_*(C,d) := \sum_{n=0}^{+\infty} \beta_n(C)$ is the homological dimension of (C,d);
- $\chi_c(C) := \sum_n (-1)^n \beta_n(C)$ is the Euler-Poincaré homological invariant of (C,d).

It is worth pointing out that the homology measures the obstruction of a bond to be a cycle. In fact, two bords x and y are homologous, i.e., $[x] = [y]$, means that $dx = dy = 0$ and that $x = y + dc$.

Definition 1.5

Let n be a fixed integer. A standard n-simplex (or standard simplex of dimension n) in $\mathbb{R}^n$, denoted generally Δ_n, is the hull convex in $\mathbb{R}^n$ of the points $e_0, e_1, \cdots, e_n$, where $e_0 = (0, \cdots, 0)$, $e_1 = (1, 0, \cdots, 0)$, $\ldots$, and $e_n = (0, \cdots, 0, 1)$.

- A 0-standard simplex is a point;
- A 1-standard simplex is a segment;
- A 2-standard simplex is a full triangle;

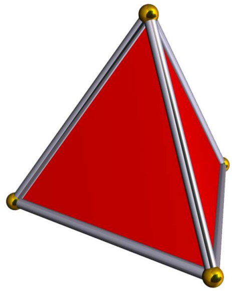

Figure 1.1: A tetrahedron is 3-simplex

- A 3-standard simplex is a full tetrahedron.

Definition 1.6

Let n be a fixed integer, and Δ a given n-standard simplex. Let $0 \leq k \leq n$. Any hull convex of a sub-family (e_i) of k elements among $e_0, e_1, \cdots, e_n$ is a k-standard simplex of Δ, called a k-face of Δ.

For example, the 0-faces of a tetrahedron are its vertices, its 1-faces are its edges, while its 2-faces are its full triangles. The table here above summarises the number of faces of some examples of n-simplices

simplex	0-faces	1-faces	2-faces	3-faces	4-faces	5-faces
Point	1	-	-	-	-	-
Segment	2	1	-	-	-	-
Triangle	3	3	1	-	-	-
Tetrahedron	4	6	4	1	-	-
Pentachord	5	10	10	5	1	-
5-simplex	6	15	20	15	6	1
6-simplex	7	21	35	35	21	7

We get this Euler-Poincaré formula:

$$\sum_{n \geq 0} (-1)^n r_n(C) = 1,$$

where r_n denotes the number of the n-faces in C.

Definition 1.7

We call a *simplicial complex* any set of simplices $\mathcal{K}$, that satisfies the following conditions:

1. Every face of a simplex from $\mathcal{K}$ is also in $\mathcal{K}$;
2. Any non empty intersection of two simplices $\sigma_1, \sigma_2 \in \mathcal{K}$ is a face of both σ_1 and σ_2.

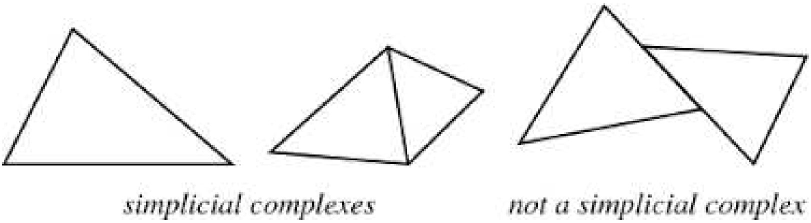

Figure 1.2: Simplicial complex or not?

Definition 1.8

Let $\mathcal{K}$ be a simplicial complex, and n a fixed integer.
We call a *n-chain* in $\mathcal{K}$ any formal sum $\sum n_i \sigma_i$, where σ_i are n-simplices in $\mathcal{K}$, with coefficients $n_i \in \mathbb{Z}$.
The subset of all this n-chains will be denoted $\mathcal{C}_n(\mathcal{K})$, with the convention that $\mathcal{C}_{-1}(\mathcal{K}) = \emptyset$.

Definition 1.9

Let $\mathcal{K}$ be a simplicial complex.
The *boundary operator* on $\mathcal{K}$, is the $\mathbb{Z}$-linear map defined by:

$$\begin{array}{rrcl} \partial_n: & \mathcal{C}_n(\mathcal{K}) & \longrightarrow & \mathcal{C}_{n-1}(\mathcal{K}) \\ \sigma: & = [e_0, \ldots, e_n] & \longmapsto & \partial_n \sigma \end{array}$$

where, $\partial_n\sigma \colon = \sum_{i=0}^{n}(-1)^i[e_0,\ldots,\hat{e}_i,\ldots,e_n]$ and that $\hat{e}_i$ means omitted.

One can check that

Theorem 1.1

$$\partial_{n-1}\circ\partial_n = 0.$$

In particular we have

$$\mathrm{Im}\partial_n \subset \ker\partial_{n-1}.$$

This yields the chain complex

$$0 \overset{i}{\hookrightarrow} C_n(\mathcal{K}) \xrightarrow{\partial_n} C_{n-1}(\mathcal{K}) \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_1} C_0(\mathcal{K}) \xrightarrow{\partial_0} 0,$$

which is an algebraic structure that consists of a sequence of abelian groups (or modules) and a sequence of homomorphisms between consecutive groups such that the image of each homomorphism is included in the kernel of the next. Elements of $\mathrm{Im}\partial_k$ are called *boundaries*, those of $\ker\partial_{k-1}$ are called *cycles*. Thus any boundary is a cycle, the inverse is not always true.

Definition 1.10

The k-th simplicial *homology group* of $\mathcal{K}$, is defined to be the quotient group

$$H_k(\mathcal{K}) = \ker\partial_{k-1}/\mathrm{Im}\partial_k.$$

Its rank, denoted $\beta_p(\mathcal{K})$, is called the k-th *Betti number* of $\mathcal{K}$.

$H_k(\mathcal{K})$ represents the obstruction of a cycle to be a boundary, and $\beta_p(\mathcal{K})$ represents the number of the homologous k-dimensional holes in a shape. Since the interior of a circle is a disc, which is a variety of dimension 1, one may consider a circle to have a one-dimensional hole. In particular β_0 is the number of the path-connected components of a shape, since two points are homotopic if and only if they live in the same path-connected component.

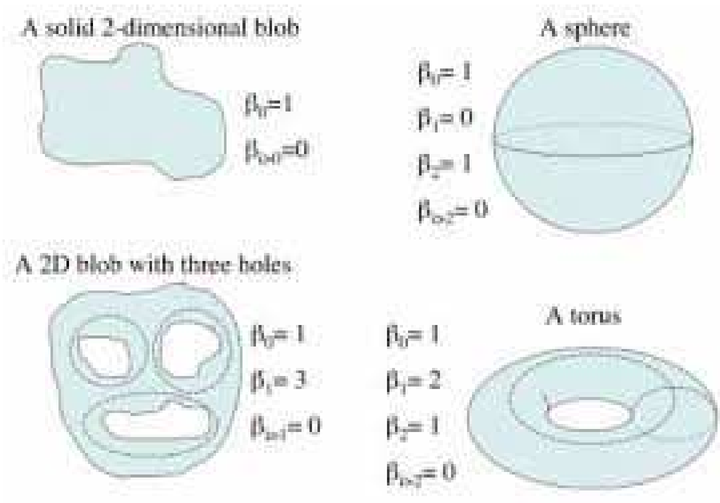

Figure 1.3: Betti numbers of some shapes.

Definition 1.11

Let X be a topological space, and n a fixed integer.
Any continuous map $\sigma\colon \Delta_n \longrightarrow X$ is called a *n-singular simplex* of X.

While identifying σ to its geometrical image in X, it is clear that:

- 0-singular simplices are points of X;
- 1-singular simplices are curves in X;
- 2-singular simplices are 3D-surfaces in X;
- 0-singular simplices are 3D-volumes in X.

Definition 1.12

Let X be a topological space, and n a fixed integer.
n-singular chains are all finite sums, $\sum n_i \sigma_i$, where n_i are integers, while σ_i are n-singular simplices.

The $\mathbb{Z}$-module of such chains, will be denoted $C_n(X)$.

Definition 1.13

Let X be a topological space, and n a fixed integer.
The bord operator $\partial\colon C_n(X) \longrightarrow C_{n-1}(X)$ is defined by

$$\partial\sigma\colon = \sum_{i=0}^{n}(-1)^i \sigma_{|\Delta^i_{n-1}},$$

where $\Delta^i_{n-1}\colon = [e_0,\dots,\hat{e}_i,\dots,e_n]$ and that $\hat{e}_i$ means omitted.

Theorem 1.2

Following the notations here above, we have

$$\partial^2 = 0$$

In particular we have

$$\mathrm{Im}\partial_n \subset \ker\partial_{n-1}.$$

Hence, we obtain a chain complex

$$0 \overset{i}{\hookrightarrow} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0,$$

whose homology is called the singular homology of X.

Definition 1.14

Let X be a topological space, and $C(X)$ its associated singular chain complex as described here above. Then, we put

$$H_*(X)\colon = H_*(C(X),\partial), \text{ the singular homology of } X.$$

Remark 1.1

Let X and Y be two given topological spaces, and $f\colon X \longrightarrow Y$ a continuous map. For any singular X-simplex $\sigma\colon \Delta_n \longrightarrow X$, one can associate a singular Y-simplex: $f \circ \sigma\colon \Delta_n \longrightarrow Y$. In particular, if σ_1 and σ_2 are two homologous cycles in X, then their images are still homologous in Y.
This enables us to define on the category **Top**, of topological spaces, endowed with continuous maps, the following invariant functor:

$$H_*\colon X \mapsto H_*(X),\ H_*\colon f \mapsto H_*(f)$$

where

$$\begin{array}{rccc} H_*(f)\colon & H_*(X) & \longrightarrow & H_*(Y) \\ & [\sigma] & \longmapsto & [f \circ \sigma] \end{array}$$

4 Cohomology

Definition 1.15

A cochain complex is any graded family $(C^n)_{n\in\mathbb{N}}$ of $\mathbb{Z}$-modules equipped with $\mathbb{Z}$-morphisms $d^n\colon C^n \longrightarrow C^{n+1}$ such that $d^{n+1} \circ d^n = 0$.

By analogy to the above, we set $B^n\colon = \mathrm{Im} d^n \subset Z^{n+1} = \ker d^{n+1}$, and define the *cohomology* of (C,d) to be

$$H^n(C,d)\colon = Z^{n+1}/B^n.$$

- Element de Z^n are called *n-cocycles*;
- elements of B_n are called *n-cobords*.

For a given topological space, X, we dualize its singular homology as follows:

$$C^n(X)\colon = C_n(X)^\#$$

and put

$$d\colon = \partial^\#,$$

we get a cochain complex $C^*(X), d)$, whose cohomolgy is called the *singular cohomology* of X.

Definition 1.16

Let X be a fixed topological space. The *cap product* is the bilinear application defined as follows:

$$\begin{array}{rrcl} \frown: & C_p(X;K)\times C^q(X;K) & \longrightarrow & C_{p-q}(X;K) \\ & (\sigma,\delta) & \longmapsto & \sigma\frown\delta :=\delta(\sigma|_{[e_0,\dots,e_q]})\sigma|_{[e_q,\dots,e_p]} \end{array},$$

that can be extended naturally to homology and cohomology as follows:

$$\frown: H_p(X;K)\times H^q(X;K)\longrightarrow H_{p-q}(X;K).$$

If X is a closed and orientable manifold of dimension n, it is well known that

$$\dim H_n(X) = 1.$$

and that

$$H_k(X) = 0, \text{for any } k > n.$$

The generator $[\mu]$ of $H_n(X;\mathbb{Q})$, called the *class fundamental* of X verifies the following:

$$[\mu]\frown[\sigma]\in H_{n-k}(X), \text{ for any } [\sigma]\in H^k(X).$$

We get the following linear application

$$\begin{array}{rrcl} D: & H^k(X) & \longrightarrow & H_{n-k}(X) \\ & [\sigma] & \longmapsto & D[\sigma] := [X]\frown[\sigma] \end{array}.$$

Theorem 1.3

Poincaré duality: If X is a closed and orientable manifold of dimension n, then the application

$$D: H^k(X;\mathbb{Q}) \longrightarrow H_{n-k}(X;\mathbb{Q}) \text{ is an isomorphism.}$$

In other words,
$$H^k(X) \cong H_{n-k}(X).$$

For further details on both homology and cohomology we suggest these standard references: [29] and [49].

5 Homotopy

Definition 1.17

Set $I\colon = [0,1]$, and let X and Y be two given topological spaces. We call a homotopy from X to Y, any continuous map:

$$H\colon X \times I \longrightarrow Y.$$

Then two continuous maps $f, g\colon X \to Y$ are said to be homotopic, when there is a homotopy $H\colon X \times I \to Y$, such that

$$H(-,0) = f, \quad H(-,1) = g,$$

then we write
$$f \sim g,$$
which define an equivalence relation on continuous maps from X to Y.

Definition 1.18

Two given topological spaces X and Y, are said to be homotopic, or have the same homotopy type if and only if there exist two continuous maps $f\colon X \to Y$ and $g : Y \to X$ such that

$$f \circ g \sim \mathrm{id}_Y, \quad g \circ f \sim \mathrm{id}_X.$$

This leads to an equivalence relation on the category **Top** of topological spaces endowed with continuous maps.

Definition 1.19

An algebraic object obj(X) associated to a topological space X, is defined to be a homotopical invariant when

$$\mathrm{obj}(X) \cong \mathrm{obj}(X),$$

for any $X \sim Y$.

Theorem 1.4

The following algebraic objects are homotopically invariant

1. The homology: $X \sim Y \Longrightarrow H_*(X) \cong H_*(Y)$;
2. Betti numbers: $X \sim Y \Longrightarrow \beta_k(X) = \beta_k(Y)$;
3. Homological and cohomological dimensions: $X \sim Y \Longrightarrow \dim H_*(X) = \dim H_*(Y)$ and $\dim H^*(X) = \dim H^*(Y)$;
4. Euler-Poincaré invariant: $X \sim Y \Longrightarrow \chi_c(X) = \chi_c(Y)$.

Definition 1.20

Let X be a path-connected topological space, and n a fixed integer.
The n-homotopy group of X is defined to be

$$\pi_n(X) \colon = \mathrm{map}(\mathbb{S}^n, X)/\sim,$$

where $\mathbb{S}^n$ denoted the unit sphere of $\mathbb{R}^{n+1}$, and $\mathrm{map}(\mathbb{S}^n, X)/\sim$ the quotient set of continuous maps $\gamma \colon \mathbb{S}^n \longrightarrow X$, up to homotopy.

Remark 1.2

With respect to the denotations above, it is worth pointing out the following:

- The homotopy groups $\pi_n(X)$ are all abelian, for $n \geq 2$;

- $\pi_0(X)$ is nothing other than the set of the path components of X;
- $\pi_1(X)$ called the fundamental group of X, describes, when loops in X are homotopic. card$(\pi_1(X))$ interprets the numbers of "holes" in X.
  In particular, if X is simply connected, then $\pi_i(X) = \{0\}$.

Definition 1.21

Let X and Y be two topological spaces, then any continuous map $f\colon X \longrightarrow Y$ can be extended naturally to the homotopy groups, by setting:

$$\begin{array}{rccc} \pi_n(f)\colon & \pi_n(X) & \longrightarrow & \pi_n(Y) \\ & [\gamma] & \longmapsto & [f \circ \gamma] \end{array}$$

f is said to be a *weak homotopy equivalence* , when all $\pi_n(f)$ are isomorphisms.
Hence X and Y are said to have the same *weak homotopy type*.

Definition 1.22

Let X and Y be two given topological spaces, A a fixed subset of X and $f: A \longrightarrow Y$ a continuous map.
We call an *attachment* of X with respect to f, the quotient set $X \cup_f Y$ obtained while identifying any element $x \in A$ with its image $f(x) \in Y$.
More precisely,

$$X \cup_f Y :=(X \coprod Y)/x \sim f(x),$$

where $X \coprod Y$ denotes the disjoint geometrical sum of X and Y.

Vocabulary: Let n be a fixed integer.

- We call n-cell, generally denoted e^n, any topological space that is homeomorphic to the open disk $D(0,1)$ of $\mathbb{R}^n$;
- We call n-skeleton, generally denoted $X^{(n)}$, any topological space that can be obtained by the attachment of $X^{(n-1)}$ to a finite number of n-cells;

- By convention, 0-skeletons, $X^{(0)}$, are any discrete collections of points.

Definition 1.23

We call a CW-complex, any topological space of the form $X = \bigcup X^{(n)}$ obtained by successive attachment of cells, and that verifies the following:

- **Closure-finite**: The boundary of each cell is equal to a disjoint union of a finite number of cells of smaller dimensions;
- **Weak topology**: If X is endowed with the weak topology, then a subset A of X is open, if and only if $A \cap X^n$ is open for any $n \in \mathbb{N}$.

The category of CW-complexes turns out to be a good category to work in homotopy, as the following results illustrate:

Definition 1.24

A continuous map $f\colon X \longrightarrow Y$ between two CW-complexes is called a *cellular map*, if it injects any n-skeleton of X into a n-skeleton of Y.
More precisely, if $f(X^{(n)}) \subset Y^{(n)}$, for any $n \in \mathbb{N}$.

Theorem 1.5

Cellular Approximation Theorem: Any continuous map between two CW-complexes is homotopic to a cellular map.

Theorem 1.6

Whitehead Theorem: Any weak homotopy equivalence between two CW-complexes is a homotopy equivalence.

Definition 1.25

We call a cellular model of a topological space X, any CW-complex that has the same weak homotopy type of X.

Theorem 1.7

Cellular Model Theorem: Any topological space has a cellular model, unique up to homotopy.

For further details on homotopy, we suggest this standard reference: [69].

Chapter 2

Rational Homotopy Theory

1 Introduction

An element of a group is said to be without torsion when its order is infinite. The group itself is said to be without torsion, or free of torsion, when its unity is the unique element with torsion. If we tensor a group G, then we obtain a $\mathbb{Q}$-vector space $G \otimes G5$, which is an abelian group without torsion. Basically, the aim of the rational homotopy theory, founded in the 1960s by D. Quillen [52] and D. Sullivan [61], is to study the rational homotopy type of a topological space by ignoring the torsion of its homotopy groups.

Definition 2.1

A topological space is said to be rational, when all its homotopy groups are $\mathbb{Q}$-vector spaces.

Theorem 2.1

If X is a simply connected CW-complex, then there exists a rational simply connected CW-complex, $X_{\mathbb{Q}}$, such that

$$\pi_n(X) \otimes \mathbb{Q} \cong \pi_n(X_{\mathbb{Q}}) \text{ as } \mathbb{Q}\text{-vector spaces.}$$

$X_{\mathbb{Q}}$ is called the *rationalization* of X, and its homotopy type is called the homotopy type of X.

In fact, the birth of rational homotopy went back a little further to 1950, when H. Hopf conjectured the following:

Conjecture 1

The homotopy type of any topological space can be modelled by a $\mathbb{Q}$-graded Lie algebra.

P. Serre was the first to study the non-torsion of the homotopy and homology groups. In 1953, he resolved the Hopf conjecture in this particular case:

Theorem 2.2

The rational weak homotopy type of any finite product of spheres of odd dimensions can be modelled by a semi-simple, compact and connected Lie group.

In 1967, D. Quillen resolved completely the Hopf conjecture in a rational context. He stated that:

Theorem 2.3

The rational homotopy type of any simply connected and pointed topological space can be modelled by a Lie group.

Theorem 2.4

The rational weak homotopy equivalence of any finite product of spheres of odd dimensions can be modelled by a semi-simple, compact and connected Lie group.

Quillen's work represented a crucial step toward the development of the rational homotopy theory, by justifying theoretically the reliability of the algebraic model as an efficient tool to determine the rational homotopy type of a topological space. However, his work suffered from a major flaw: the calculations were generally difficult or even impossible.

In the early 1970s, D. Sullivan tackled this problem of calculations, and proposed a model dual to that of Quillen. Sullivan's model is a co-chain of commutative algebras, based on piecewise linear rational forms. When publishing his first results, Sullivan pointed to the possibility of applying his models to resolve some geometric problems, such as the study of non-abelian periods in a differential manifold.

He claims that:

> Any reasonable geometric construction on a topological space can be reflected by another finite, algebraic, using minimal models.

2 Hilali Conjecture

We were interested especially in some open problems related to elliptic spaces: the topological spaces X, whose rational homotopy $\pi_*(X)\otimes\mathbb{Q}$, and rational homology $H_*(X;\mathbb{Q})$ are both of finite dimension. Around 2007, our research focused on the following open problem:

Conjecture 2

Hilali conjecture (Topological version), [32]: For any simply connected elliptic space X, we have:

$$\dim H^*(X;\mathbb{Q}) \geq \dim(\pi_*(X)\otimes\mathbb{Q}).$$

One of the powerful tools we used from rational homotopy theory, was the Sullivan minimal model, which relates by a homotopy equivalence the category of simply connected topological spaces to that of commutative differential graded algebras. This allows topologists to transpose many of their topological problems in a algebraic version, as follows:

Theorem 2.5

Sullivan [61]: For any simply connected topological space X of finite type, i.e.,
$\dim H^k(X;\mathbb{Q}) < \infty$ for all $k > 0$, there exists a commutative differential graded algebra $(\Lambda V, d)$, called the *minimal Sullivan model* of X, which algebraically models the rational homotopy of X, in the sense that

$$\pi_n(X)\otimes\mathbb{Q} \cong V \text{ as vector spaces,}$$

and that

$$H_*(X;\mathbb{Q}) \cong H_*(\Lambda V, d) \text{ as algebras.}$$

That means that any simply-connected topological space X, can be replaced by a rational CW-complex $X_{\mathbb{Q}}$, without exchanging either the rational homotopy type, or the rational cohomology. In particular, we get:

$$\begin{aligned} \dim H_*(X;\mathbb{Q}) &= \dim H_*(\Lambda V, d); \\ \dim \pi_n(X) &= \dim V, \end{aligned}$$

and the

Conjecture 3

Hilali conjecture (Algebraic version), [32]: If $(\Lambda V, d)$ is a simply connected and elliptic model of Sullivan, then

$$\dim V \leqslant \dim H^*(\Lambda V, d).$$

Before our works, the conjecture holds uniquely for pure models ($\dim V^{\text{even}} = \dim V^{\text{even}}$), [32]:

Theorem 2.6

If $(\Lambda V, d)$ is a simply connected and elliptic pure model of Sullivan , then
$$\dim V \leqslant \dim H^*(\Lambda V, d).$$

From 2008, we proved the following:

Theorem 2.1

Hilali & M. (2008), [36]: The Hilali conjecture holds for H-spaces.

Let us recall that H-spaces are topological spaces whose Sullivan models are of the form $(\Lambda V, d)$. Topological groups are particular examples of H-spaces.

Theorem 2.2

Hilali & M. (2008), [36]: The Hilali conjecture holds for simply connected and elliptic topological spaces X, such that
$$\mathrm{fd}(X) \leq 10,$$
where $\mathrm{fd}(X) := \max\{k \in \mathbb{N}, \dim H^k(X; \mathbb{Q}) \neq 0.$

Theorem 2.3

Hilali & M. (2008), [36]: The Hilali conjecture holds for simply connected and hyper-elliptic minimal Sullivan models, under some restrictive conditions.

Let us recall that a minimal Sullivan model $(\Lambda V, d)$ is called hyper-elliptic whenever it satisfies the following:
$$dV^{\text{even}} = 0 \text{ and } dV^{\text{odd}} \subset \Lambda V^{\text{even}} \otimes \Lambda V^{\text{odd}}.$$
Pure models are particular examples of hyper-elliptic under our restrictive conditions.

Theorem 2.4

Hilali & M. (2008), [36]: The Hilali conjecture holds for symplectic manifolds, for cosymplectic manifolds, and for nilmanifolds.

We especially show that for such manifolds, the inequality in the Hilali conjecture is strict.

Theorem 2.5

Hilali & M. (2008), [37]: The Hilali conjecture holds for simply connected and elliptic formal topological spaces.

Formal spaces are topological spaces whose minimal Sullivan models $(\Lambda V, d)$ verify the following:

$$V = U \oplus W, \text{ with } dV = 0, \text{ and } dW \text{ is a regular sequence in } \Lambda U.$$

Examples of formal spaces include spheres, H-spaces, symmetric spaces, and compact Kähler manifolds.

Theorem 2.6

Hilali & M. (2008), [37]: The Hilali conjecture holds for simply connected and elliptic minimal Sullivan models $(\Lambda V, d)$ whose differential is homogeneous of length at least 3, i.e. $dV \subset \Lambda^{\geq 3} V$.

Around 2014, we investigated the case of coformal spaces, those for whom the differential of the Sullivan model is purely quadratic, i.e., $dV \subset \Lambda^2 V$. We especially prove the following:

Theorem 2.7

Elkrafi, Hilali & M. (2015), [13]: The Hilali conjecture holds for any coformal space X whose rational homotopy Lie algebra $\mathbb{L}$ is of nilpotency 1 or 2.

We also proposed some research directions to resolve completely the coformal case by induction on the nilpotency degree of the associated homotopy Lie algebra. In fact, resolving completely the coformal case would be a decisive step towards the definitive resolution of the Hilali conjecture, since the case when the differential is homogeneous of length at least 3 was already resolved. The Hilali conjecture now belongs to the rational homotopy theory folkloric open problems, and it gave rise to a lot of research interests and is now stated in many interesting cases, such as:

- Hyper elliptic Sullivan models, see [4];
- Two stages Sullivan models, i.e. when $V = U \oplus W$ with $dU = 0$ and $dW \subset \Lambda^{\geq 2} U$, see [1].

We also investigate the Hilali conjecture for configuration spaces of manifolds. Let us recall that

Definition 2.2

If M is given a manifold and k a fixed non null integer, then

$$F(M,k) = \{(x_1, x_2, \ldots, x_k) \in M^k \ , \ x_i \neq x_j \text{ for } i \neq j\}$$

denotes the space of all ordered configurations of k distinct points in M.

Our main result states that

Theorem 2.8

Hilali, M. and Yamoul (2015), [39]: If M is a closed and simply connected manifold, then $F(M,k)$ verifies the Hilali conjecture provided that $F(M,k)$ is elliptic.

We also proved the following

Theorem 2.9

Hilali, M. and Yamoul (2015), [39]: If M is rationally elliptic, and $X = M - \{pt\}$ has a non-trivial rational homotopy group in dimension > 1, then $F(X,2)$ and $F(M,k)$ for $k > 2$, are rationally

- hyperbolic.

Theorem 2.10

Hilali, M. and Yamoul (2015), [39]: If M is a simply connected manifold of dimension at least 3, and has at least two linearly independent elements in its rational cohomology, then $F(M,3)$ and in general $F(M,k), k \geq 3$ is rationally hyperbolic.

where a topological space is said to be hyperbolic, whenever its homology is of infinite dimension

3 Halperin Conjecture

Around 2016 we were especially interested in:

Conjecture 4

Halperin conjecture [30]: For any elliptic space X, we have:

$$\dim H^*(X;\mathbb{Q}) \geq 2^{\mathrm{rk}_0(X)},$$

where $\mathrm{rk}_0(X)$, called the toral rank, is defined to be the maximum, or the infinity, of integers n such that the toral $\mathbb{T}^n$ acts almost freely on X.

We firstly make the connection possible between this conjecture and that of Hilali, thanks to the following results:

Let χ_c and χ_π be the cohomological and homotopic Euler-Poincaré characteristics of X, respectively defined by:

$$\begin{aligned}
\chi_c: &= \sum_{k\geq 0} (-1)^k \dim H^k(X;\mathbb{Q}); \\
\chi_\pi: &= \sum_{k\geq 0} (-1)^k \dim \pi_n(X) \otimes \mathbb{Q}.
\end{aligned}$$