# Post-Newtonian Hydrodynamics

# Post-Newtonian Hydrodynamics:

## *Theory and Applications*

By

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*To Maria Rachel*

# Contents

# PREFACE

This book is about the post-Newtonian theory, a method of successive approximations of Einstein's field equations in powers of the light speed. This method was proposed in 1938 by Einstein, Infeld and Hoffmann[1] and in 1965 the first post-Newtonian hydrodynamic equations for a perfect fluid were derived by Chandrasekhar.[2] Nowadays the post-Newtonian theory is still a field of investigation by many researches.

The aim of this book is to present the post-Newtonian theory and some applications in a self-contained manner. The development of the theory follows the works of Chandrasekhar and its collaborators and the book by Weinberg.[3] For another different approach and applications of the post-Newtonian theory

[1] A. Einstein, L. Infeld and B. Hoffmann, The gravitational equations and the problem of motion, *Ann. of Math.* **39**, 65 (1938).

[2] S. Chandrasekhar, The post-Newtonian equations of hydrodynamics in general relativity, *Ap. J.* **142**, 1488 (1965).

[3] S. Weinberg, *Gravitation and cosmology. Principles and applications of the theory of relativity* (Wiley, New York, 1972).

the reader is referred to the book by Poisson and Will.[4]

The book is organized as follows. In the first Chapter an overview of the non-relativistic and relativistic Boltzmann equation with the corresponding transfer and balance equations are introduced. The particle four-flow and the energy-momentum tensor are calculated with the equilibrium Maxwell-Jüttner distribution function and it is shown that the equilibrium condition of the Boltzmann equation in gravitational fields leads to Tolman and Klein laws.

In Chapter two the first post-Newtonian approximation of Einstein's field equations is determined from Chandrasekhar and Weinberg methods, which introduce different gauge conditions and equivalent gravitational potentials. The post-Newtonian balance equations for an Eulerian and non-perfect fluids are obtained and the Brans-Dicke theory in the post-Newtonian approximation is developed. Other subjects of this chapter include the analysis of the gravitational potentials, the conservation laws and the virial theorem in the post-Newtonian approximation.

The second post-Newtonian approximation is the subject of Chapter three, where new gravitational potentials come out from Einstein's field equations. The Eulerian balance equations are determined and the conservation laws are investigated in this approximation.

In Chapter four the first and second post-Newtonian approximations of the Boltzmann equation and of the Maxwell-Jüttner

[4]E. Poisson and C. M. Will, *Gravity: Newtonian, Post-Newtonian, Relativistic*, (Cambridge UP, Cambridge, 2014).

distribution function are derived. From a transfer equation of the post-Newtonian Boltzmann equations the Eulerian balance equations for perfect gases are obtained for the two approximations. Furthermore, the post-Newtonian Jeans equations for stationary spherically symmetrical and axisymmetrical self-gravitating systems are derived.

The aim of Chapter five is the search for polytropic solutions of the post-Newtonian Lane-Emden equation for some stars like the Sun, white and brown dwarfs, red giants and neutron stars. The post-Newtonian solutions are compared with the ones that come out from the Newtonian Lane-Emden equation.

In Chapter six the problem of spherically symmetrical accretion is investigated where the Bernoulli equation and the critical values of the flow fields are determined in the post-Newtonian approximation. The solutions of the post-Newtonian Bernoulli equation are compared with the ones that follow from the Bernoulli equations of a relativistic theory and its weak field approximation.

The Jeans instability from the hydrodynamic equations is the subject of Chapter seven. Here the Newtonian Jeans instability is investigated for a non-expanding and expanding Universe. The post-Newtonian Jeans instability are obtained from the mass density and momentum density balance equations in the first and second approximations.

The aim of Chapter eight is to study Jeans instability within the framework of the Boltzmann equation. For the Newtonian and post-Newtonian Boltzmann equations two approaches are used to obtain the dispersion relation which leads to the Jeans instability. In one of them the perturbed distribution function is left unspecified while in the other the perturbed distribution function is written in terms of the summational invariants of the Boltzmann equation. The determination of Jeans instability for an expanding Universe and for a BGK model of the Boltzmann equation – where collision between the particles are taken into account – are also examined.

In the last chapter it is investigated the rotation curves of galaxies within the post-Newtonian framework and the solution of Jeans equation for stationary spherically symmetrical self-gravitating systems.

The notations used in this book are: Greek indices take the values 0,1,2,3 and Latin indices the values 1,2,3. The semicolon denotes the covariant differentiation, the indices of Cartesian tensors will be written as subscripts, the summation convention over repeated indices will be assumed and the partial differentiation will be denoted by $\partial/\partial x^i$.

It is expected that this book can be helpful not only as a text for advanced courses but also as a reference for physicists, astrophysicists and applied mathematicians who are interested in the post-Newtonian theory and its applications.

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# CHAPTER 1

# THE BOLTZMANN EQUATION: AN OVERVIEW

In this chapter an outline of the Boltzmann equation is presented. The non-relativistic Boltzmann equation is based on the book [1] while the relativistic one on the book [2]. For more details and references on non-relativistic and relativistic Boltzmann equation the reader should consult these two books and the references therein.

## 1.1 Non-relativistic Boltzmann equation

The Boltzmann equation is a non-linear integro-differential equation for the space-time evolution of the one-particle distribution function $f(\mathbf{x}, \mathbf{v}, t)$ in the phase space spanned by the space coordinates $\mathbf{x}$ and velocity $\mathbf{v}$ of the particles. The one-particle distribution function is such that $dN = f(\mathbf{x}, \mathbf{v}, t)d^3xd^3v$ gives at time $t$ the number of particles in the volume element $d^3x$ about $\mathbf{x}$ and with velocities in a range $d^3v$ about $\mathbf{v}$. In the non-relativistic kinetic theory of monatomic gases the Boltzmann equation reads

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} + F_i \frac{\partial f}{\partial v_i} = \int \big[f(\mathbf{x}, \mathbf{v}'_*, t) f(\mathbf{x}, \mathbf{v}', t) - f(\mathbf{x}, \mathbf{v}_*, t) f(\mathbf{x}, \mathbf{v}, t)\big] g \, \sigma \, d\Omega \, d^3 v_*. \tag{1.1}$$

Here $\mathbf{F}$ is a force per unit mass which acts on the particles and do not depend on its velocities. The right-hand side is a consequence of the so-called *Stoßzahlansatz* which considers only binary collisions of two beams of particles which before collision have velocities $(\mathbf{v}, \mathbf{v}_*)$ and after collision $(\mathbf{v}', \mathbf{v}'_*)$. Furthermore, $g = |\mathbf{v}_* - \mathbf{v}|$ is a relative velocity, $\sigma$ a collision differential cross section and $d\Omega$ an element of solid angle of the scattered particles. In the binary collision the momentum and energy conservation laws hold

$$m\mathbf{v} + m\mathbf{v}_* = m\mathbf{v}' + m\mathbf{v}'_*, \quad \frac{1}{2}mv^2 + \frac{1}{2}mv_*^2 = \frac{1}{2}mv'^2 + \frac{1}{2}mv'^2_*, \tag{1.2}$$

where $m$ is the particle rest mass.

In the kinetic theory of gases the macroscopic fields are given in terms of integrals over the microscopic quantities of the particles multiplied by the one-particle distribution function. The microscopic quantities mass $m$, momentum $m\mathbf{v}$ and energy $mv^2/2$ of a particle imply the macroscopic fields of mass density $\rho$, momentum density $\rho\mathbf{V}$ and energy density $\rho u$ of the gas defined by

$$\rho(\mathbf{x},t) = \int mf(\mathbf{x},\mathbf{v},t)d^3v, \;\; \rho\mathbf{V}(\mathbf{x},t) = \int m\mathbf{v}f(\mathbf{x},\mathbf{v},t)d^3v, \quad (1.3)$$

$$\rho u(\mathbf{x},t) = \int \frac{m}{2}v^2 f(\mathbf{x},\mathbf{v},t)d^3v. \quad (1.4)$$

The energy density can be decomposed into a sum of a kinetic energy density $\rho V^2/2$ and an internal energy density $\rho\varepsilon$ by introducing the peculiar velocity $\mathcal{V}_i = v_i - V_i$ which is the difference of the particle velocity $\mathbf{v}$ and the hydrodynamic velocity $\mathbf{V}$. Hence we have

$$\rho u = \frac{1}{2}\rho V^2 + \rho\varepsilon, \quad \text{where} \quad \rho\varepsilon = \int \frac{1}{2}m\mathcal{V}^2 f(\mathbf{x},\mathbf{v},t)d^3v. \quad (1.5)$$

Note that $\int \mathcal{V}_i f d^3v = 0$.

An important quantity in the kinetic theory of gases is the so-called summational invariant $\psi$ defined by the relationship $\psi + \psi_* = \psi' + \psi'_*$. It is easy to see that the mass $m$, the momentum $m\mathbf{v}$ and the energy $mv^2/2$ of a particle are summational invariants. One important consequence is that the representation of the summational invariant as a sum of mass, momentum and energy of a particle leads to the determination of the one-particle distribution function at equilibrium. Indeed, the equilibrium is characterized when the collision term of the Boltzmann equation (1.1) vanishes, i.e., at equilibrium the number of particles entering in the phase space volume is equal to those that leaving it. In this sense $f(\mathbf{x}, \mathbf{v}'_*, t)f(\mathbf{x}, \mathbf{v}', t) = f(\mathbf{x}, \mathbf{v}_*, t)f(\mathbf{x}, \mathbf{v}, t)$ implying that $\ln f(\mathbf{x}, \mathbf{v}, t)$ is a summation invariant so that at equilibrium the one-particle distribution function becomes the Maxwellian distribution function

$$f = \frac{\rho}{m}\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \exp\left[-\frac{m\mathcal{V}^2}{2kT}\right], \tag{1.6}$$

where the absolute temperature $T$ is related with the specific internal energy by $\varepsilon = 3kT/2m$ with $k$ denoting the Boltzmann constant.

The derivation of hydrodynamic equations from a transfer equation for arbitrary macroscopic quantities which are associated with mean values of microscopic quantities is an old subject in the literature of kinetic theory of gases which goes back to the work of Maxwell in 1867 [3]. In 1911 Enskog [4] determined from the Boltzmann equation a general transfer equation for an arbitrary function of the space-time and particle velocity where the hydrodynamic equations could be obtained. The starting point for the knowledge of the so-called Maxwell-Enskog transfer equation follows from the multiplication of the Boltzmann equation (1.1) by an arbitrary function of the space-time coordinates and particle velocity $\Psi(\mathbf{x}, \mathbf{v}, t)$ and subsequent integration of the resulting equation over all values of the particle velocity components $d^3v$. Hence it follows the Maxwell-Enskog transfer equation

$$\begin{aligned}\int \Psi \left[\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} + F_i \frac{\partial f}{\partial v_i}\right] d^3v = \frac{\partial}{\partial t} \int \Psi f d^3v \\ + \frac{\partial}{\partial x_i} \int \Psi v_i f d^3v + \underline{\int \frac{\partial \Psi f F_i}{\partial v_i} d^3v} \\ - \int \left[\frac{\partial \Psi}{\partial t} + v_i \frac{\partial \Psi}{\partial x_i} + F_i \frac{\partial \Psi}{\partial v_i}\right] f d^3v \\ = \frac{1}{4} \int [\Psi + \Psi_* - \Psi' - \Psi'_*] \, [f'_* f' - f_* f] \, g \, \sigma \, d\Omega \, d^3v_* d^3v. \end{aligned} \tag{1.7}$$

In the above equation the underlined term vanishes since it can be converted by the use of the divergence theorem into an integral over a surface situated far away in the velocity space where the distribution function tends to zero. Its right-hand side follows by considering the symmetry properties of the collision operator of the Boltzmann equation where it was introduced the abbreviations $f'_* \equiv f(\mathbf{x}, \mathbf{v}'_*, t)$, $f \equiv f(\mathbf{x}, \mathbf{v}, t)$ and so on. Note that the right-hand side of the transfer equation vanishes if $\Psi$ is a summational invariant, i.e., for $\Psi \equiv \psi$.

The balance equations for the fields of mass density $\rho$, momentum density $\rho\mathbf{V}$ and energy density $\rho u$ are obtained from the transfer equation (1.7) by choosing $\Psi$ equal to the mass $m$, momentum $m\mathbf{v}$ and energy $mv^2/2$ of the particles. Hence, it follows respectively

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho V_i}{\partial x_i} = 0, \tag{1.8}$$

$$\frac{\partial \rho V_i}{\partial t} + \frac{\partial(\rho V_i V_j + p_{ij})}{\partial x_j} = -\rho \frac{\partial \phi}{\partial x_i}, \tag{1.9}$$

$$\frac{\partial \left[\rho\left(\varepsilon + \frac{V^2}{2}\right)\right]}{\partial t} + \frac{\partial \left[\rho\left(\varepsilon + \frac{V^2}{2}\right) V_i + q_i + p_{ij} V_j\right]}{\partial x_i} = -\rho \frac{\partial \phi}{\partial x_i} V_i. \tag{1.10}$$

In the above equations we have identified the force per unit mass $\mathbf{F}$ as the gravitational field $\mathbf{g} = -\nabla\phi$ where $\phi$ is the Newtonian gravitational potential, which is related with the mass density $\rho$ and the universal gravitational constant $G$ through the Poisson equation $\nabla^2\phi = 4\pi G\rho$. Furthermore, it was introduced the pressure tensor $p_{ij}$ and the heat flux vector $q_i$ which are given in terms of the one-particle distribution function by

$$p_{ij} = \int m\mathcal{V}_i\mathcal{V}_j f d^3v, \qquad q_i = \int \frac{1}{2} m\mathcal{V}^2\mathcal{V}_i f d^3v. \tag{1.11}$$

The pressure is the trace of the pressure tensor $p = p_{rr}/3$ and for perfect gases it is related to the specific internal energy by $p = 2\rho\varepsilon/3 = \rho kT/m$.

If we eliminate the time derivative of the hydrodynamic velocity $\mathbf{V}$ from the balance equation for the energy density (1.10) by using the momentum density balance equation (1.9) we get the internal energy density balance equation

$$\frac{\partial \rho\varepsilon}{\partial t} + \frac{\partial(\rho\varepsilon V_i + q_i)}{\partial x_i} + p_{ij}\frac{\partial V_i}{\partial x_i} = 0. \tag{1.12}$$

## 1.2 Boltzmann equation in special relativity

In special relativity it is considered that a gas particle of rest mass $m$ is characterized by the space-time coordinates $(x^\alpha) = (ct, \mathbf{x})$ and momentum four-vector $(p^\alpha) = (p^0, \mathbf{p})$. From the constraint that the length of the momentum four-vector is equal to $mc$, its time component $p^0$ is given in terms of the spatial components $\mathbf{p}$ by $p^0 = \sqrt{|\mathbf{p}|^2 + m^2c^2}$.

The one-particle distribution function $f(x^\alpha, p^\alpha) = f(\mathbf{x}, \mathbf{p}, t)$ is defined in terms of the space-time and momentum coordinates so that the number of particles in the volume element $d^3x$ about $\mathbf{x}$ and with momenta in a range $d^3p$ about $\mathbf{p}$ at time $t$ is given by $dN = f(\mathbf{x}, \mathbf{p}, t)d^3x\, d^3p$.

In order to know if the one-particle distribution function is a scalar invariant we have to know if $d^3xd^3p$ is a scalar invariant, because the number of particles in a volume element is indeed a scalar invariant due to fact that all observers will count the same number of particles.

We consider two inertial systems which transform according a homogeneous Lorentz group in a Minkowski space-time and whose components of the metric tensor are $\mathrm{diag}(1, -1, -1, -1)$. The volume elements $d^4x = d^4x'$ and $d^4p = d^4p'$ are scalar invariants. If we choose the primed frame of reference as a rest frame where $\mathbf{p}' = \mathbf{0}$, we have that $d^3x'$ is the proper volume whose transformation law is

$$d^3x = \sqrt{1 - v^2/c^2}d^3x'. \tag{1.13}$$

The transformation law for $p^0$ and $d^3p$ – by taking into account the primed frame as a rest frame where $\mathbf{p}' = \mathbf{0}$ – are

$$p^0 = \frac{1}{\sqrt{1-v^2/c^2}} p'^0, \qquad \frac{d^3p'}{p'_0} = \frac{d^3p}{p_0}. \tag{1.14}$$

In a Minkowski space-time $p_0 = p^0$ hence from the above equations we have that $d^3xd^3p = d^3x'd^3p'$ is a scalar invariant and as a consequence the one-particle distribution function is also a scalar invariant. Note that $d^3p/p_0$ is a scalar invariant.

In the phase space spanned by the space coordinates $\mathbf{x}$ and momentum $\mathbf{p}$ of the particles the space-time evolution of the one-particle distribution function $f(\mathbf{x}, \mathbf{p}, t)$ is given by the Boltzmann equation

$$p^\mu \frac{\partial f}{\partial x^\mu} = \int \Big[ f(\mathbf{x}, \mathbf{p}'_*, t) f(\mathbf{x}, \mathbf{p}', t) - f(\mathbf{x}, \mathbf{p}_*, t) f(\mathbf{x}, \mathbf{p}, t) \Big] F\, \sigma\, d\Omega \frac{d^3p_*}{p_{*0}}. \tag{1.15}$$

The right-hand side of the above equation represents the collision term which takes into account the binary collision of two beams of particles which before collision have momenta $(\mathbf{p}, \mathbf{p}_*)$ and after collision $(\mathbf{p}', \mathbf{p}'_*)$. The relative velocity here is given by the invariant flux

$$F = \frac{p^0 p^0_*}{c} \sqrt{(\mathbf{v} - \mathbf{v}_*)^2 - \frac{1}{c^2}(\mathbf{v} \times \mathbf{v}_*)^2} = \sqrt{(p^\alpha_* p_\alpha)^2 - m^4 c^4}. \tag{1.16}$$

Furthermore, $\sigma$ is the invariant differential cross-section and $d\Omega$ the solid angle element. At collision the energy-momentum conservation law holds

$$p^\mu + p_*^\mu = p'^\mu + p_*'^\mu, \tag{1.17}$$

which is a summational invariant.

The transfer equation for an arbitrary function $\Psi(x^\mu, p^\mu)$ is obtained from the multiplication of the Boltzmann equation (1.15) by $\Psi(x^\mu, p^\mu)$ and integration of the resulting equation with respect to $d^3p/p_0$, yielding

$$\frac{\partial}{\partial x^\mu}\int \Psi p^\mu f \frac{d^3p}{p_0} - \int \frac{\partial \Psi}{\partial x^\mu} p^\mu f \frac{d^3p}{p_0}$$
$$= \frac{1}{4}\int \left[\Psi + \Psi_* - \Psi' - \Psi_*'\right]\left[f_*' f' - f_* f\right] F\, \sigma\, d\Omega \frac{d^3p_*}{p_{*0}}\frac{d^3p}{p_0}, \tag{1.18}$$

where the right-hand side follows from the symmetry properties of the collision operator of the Boltzmann equation. Here it was introduced the abbreviations $f_*' \equiv f(\mathbf{x}, \mathbf{p}_*', t)$, $f \equiv f(\mathbf{x}, \mathbf{p}, t)$ and so on.

The equilibrium state is attained when the right-hand side of Boltzmann equation (1.15) vanishes so that $\ln f(\mathbf{x}, \mathbf{p}, t)$ is a summational invariant and the one-particle distribution function at equilibrium becomes the Maxwell-Jüttner distribution function

$$f(\mathbf{x}, \mathbf{p}, t) = \frac{n}{4\pi m^2 ckTK_2(\zeta)} \exp\left(-\frac{p^\mu U_\mu}{kT}\right). \tag{1.19}$$

Here $n$ is the particle number density, $U_\mu$ the hydrodynamic four-velocity – such that $U_\mu U^\mu = c^2$ – and $K_2(\zeta)$ the modified Bessel function of second kind defined by

$$K_n(\zeta) = \left(\frac{\zeta}{2}\right)^n \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)} \int_1^\infty e^{-\zeta y}(y^2-1)^{n-\frac{1}{2}}dy. \quad (1.20)$$

The relativistic parameter $\zeta = mc^2/kT$ is the ratio of the rest energy of the gas particle $mc^2$ and the thermal energy of the gas $kT$. In the non-relativistic limiting case $\zeta \gg 1$ while in the ultra-relativistic limiting case $\zeta \ll 1$.

The macroscopic fields of particle four-flow $N^\mu$ and energy-momentum tensor $T^{\mu\nu}$ are defined in terms of the one-particle distribution function as

$$N^\mu = \int cp^\mu f(\mathbf{x},\mathbf{p},t)\frac{d^3p}{p_0},\ T^{\mu\nu} = \int cp^\mu p^\nu f(\mathbf{x},\mathbf{p},t)\frac{d^3p}{p_0}. \quad (1.21)$$

The balance equations for the macroscopic fields are obtained from the transfer equation (1.18) by choosing $\Psi = c$ and $\Psi = cp^\mu$, yielding

$$\frac{\partial}{\partial x^\mu}\int cp^\mu f\frac{d^3p}{p_0} = 0, \quad \Rightarrow \quad \partial_\mu N^\mu = 0, \quad (1.22)$$

$$\frac{\partial}{\partial x^\nu}\int cp^\mu p^\nu f\frac{d^3p}{p_0} = 0, \quad \Rightarrow \quad \partial_\nu T^{\mu\nu} = 0. \quad (1.23)$$

Let us determine the equilibrium values of the particle four-flow $N^\mu$ and energy-momentum tensor $T^{\mu\nu}$ from the Maxwell-Jüttner distribution function. We choose a local Lorentz frame where the spatial components of the hydrodynamic four-velocity vanishes, i.e., $U^\mu = (c, \mathbf{0})$ and write the particle four-flow as

$$N^\mu = \int c p^\mu f \frac{d^3p}{p_0} = -\frac{cn}{4\pi m^2 ckTK_2(\zeta)} \frac{\partial}{\partial \mathcal{U}_\mu} \int e^{-(p^\mu \mathcal{U}_\mu)} \frac{d^3p}{p_0}, \tag{1.24}$$

where we have introduced $\mathcal{U}_\mu = U_\mu/kT$ which obeys the relationships

$$\mathcal{U}^\mu \mathcal{U}_\mu = \frac{\zeta^2}{(mc)^2}, \qquad \frac{\partial \zeta}{\partial \mathcal{U}_\mu} = \frac{(mc)^2}{\zeta} \mathcal{U}^\mu = mU^\mu. \tag{1.25}$$

In a local Lorentz frame we can use spherical coordinates to write

$$d^3p = |\mathbf{p}|^2 \sin\theta d|\mathbf{p}| d\theta d\varphi, \tag{1.26}$$

where the range of the angles are $0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$. Furthermore we change the integration variable and introduce a new variable $y$ such that

$$p_0 = mcy, \quad |\mathbf{p}|^2 = p_0^2 - m^2c^2 = m^2c^2(y^2 - 1), \tag{1.27}$$

$$\frac{d|\mathbf{p}|}{p_0} = \frac{dy}{\sqrt{y^2 - 1}}. \tag{1.28}$$

Hence by considering that the integrals over the angles $\theta$ and $\varphi$ furnish $4\pi$, (1.24) becomes

$$N^\mu = -\frac{\zeta n}{mK_2(\zeta)} \frac{\partial}{\partial \mathcal{U}_\mu} \int e^{-\zeta y} \sqrt{y^2 - 1} dy$$
$$= -\frac{\zeta n}{mK_2(\zeta)} \frac{\partial K_1(\zeta)/\zeta}{\partial \mathcal{U}_\mu} = nU^\mu. \tag{1.29}$$

The evaluation of the energy-momentum tensor proceeds in the same way

$$T^{\mu\nu} = \int cp^\mu p^\nu f \frac{d^3p}{p_0} = \frac{\zeta n}{mK_2(\zeta)} \frac{\partial^2 K_1(\zeta)/\zeta}{\partial \mathcal{U}_\mu \partial \mathcal{U}_\nu}$$
$$= (\epsilon + p)\frac{U^\mu U^\nu}{c^2} - pg^{\mu\nu}, \tag{1.30}$$

Here $g^{\mu\nu}$ is the Minkowski metric tensor. The energy density $\epsilon$ and the hydrostatic pressure $p$ are given by

$$\epsilon = \rho c^2 \left(\frac{K_3(\zeta)}{K_2(\zeta)} - \frac{1}{\zeta}\right), \qquad p = nkT. \tag{1.31}$$

In the above equations it was used the recurrence relation for the modified Bessel function of second kind

$$\frac{d}{d\zeta}\left(\frac{K_n(\zeta)}{\zeta^n}\right) = -\frac{K_{n+1}}{\zeta^n}. \tag{1.32}$$

The energy density has the following values in the non-relativistic $\zeta \gg 1$ and ultra-relativistic $\zeta \ll 1$ limiting cases

$$\epsilon = \rho c^2 \left(1 + \frac{3kT}{2mc^2}\right), \qquad \text{for} \qquad \zeta \gg 1, \tag{1.33}$$
$$\epsilon = 3nkT = 3p, \qquad \text{for} \qquad \zeta \ll 1, \tag{1.34}$$

by using the asymptotic expressions for the modified Bessel function of the second kind given in the Appendix.

Another quantity that is very important in the analysis of the Boltzmann equation is the entropy. In a relativistic theory the entropy four-flow is given in terms of the one-particle distribution function by

$$S^\mu = -k \int cf \ln f \, p^\mu \frac{d^3p}{p_0}. \tag{1.35}$$

If we choose $\Psi = -kc \ln f$ in the transfer equation (1.18) we get the balance equation for the entropy four-flow

$$\frac{\partial}{\partial x^\mu} \int (-kc \ln f) p^\mu f \frac{d^3p}{p_0} = -kc \int \frac{\partial f}{\partial x^\mu} p^\mu f \frac{d^3p}{p_0}$$
$$+ \frac{kc}{4} \int \left[ \ln \frac{f' f'_*}{f f_*} \right] \left[ \frac{f' f'_*}{f f_*} - 1 \right] f_* f \, F \, \sigma \, d\Omega \frac{d^3p_*}{p_{*0}} \frac{d^3p}{p_0}. \tag{1.36}$$

The first term in the right-hand side of the above equation vanishes, since it can be identified as the multiplication of the Boltzmann equation (1.15) by $kc$, integration over all values of $\frac{d^3p}{p_0}$ and considering the symmetry properties of the collision operator. The second term is non-negative thanks to the relationship $(x-1)\ln x \geq 0$ which is valid for all $x > 0$. Hence the entropy four-flow balance equation reduces to

$$\partial_\mu S^\mu \geq 0. \tag{1.37}$$

The equilibrium entropy four-flow can be obtained from the insertion of the Maxwell-Jüttner distribution function (1.19) into its definition (1.35) and integration of the resulting equation, yielding

$$S^\mu = k \int f p^\mu e^{-\frac{p^\mu U_\mu}{kT}} \left\{ \frac{p^\nu U_\nu}{kT} - \ln \left[ \frac{n}{4\pi m^2 ckTK_2(\zeta)} \right] \right\} \frac{d^3p}{p_0}$$

$$= \frac{T^{\mu\nu}U_\nu}{T} + k\ln\left[\frac{4\pi m^2 ckTK_2(\zeta)}{n}\right]N^\mu$$
$$= n\left\{k\ln\left[\frac{4\pi m^2 ckTK_2(\zeta)}{n}\right] + \frac{\epsilon}{nT}\right\}U^\mu. \tag{1.38}$$

thanks to (1.29) and (1.30). The entropy per particle $s$ is related to the equilibrium value of the entropy four-flow written as $S^\mu = nsU^\mu$.

The Gibbs function per particle is identified with the chemical potential $\mu$ and defined by

$$\mu = \frac{\epsilon}{n} - Ts + \frac{p}{n} = kT\left\{\ln\left[\frac{n}{4\pi m^2 ckTK_2(\zeta)}\right] + 1\right\}. \tag{1.39}$$

From this last result we can rewrite the Maxwell-Jüttner distribution function (1.19) as

$$f = \exp\left[\frac{\mu}{kT} - 1 - \frac{p_\mu U^\mu}{kT}\right]. \tag{1.40}$$

## 1.3 Boltzmann equation in gravitational fields

In order to write the number of particles in terms of the one-particle distribution function we have to know the transformations of the volume elements $d^3x$ and $d^3p$ in a Riemannian space. These transformations read

$$p^0\sqrt{-g}d^3x = p'^0\sqrt{-g'}d^3x', \qquad \sqrt{-g}\frac{d^3p}{p_0} = \sqrt{-g'}\frac{d^3p'}{p'_0}, \tag{1.41}$$