A Concise Approach to Dynamics

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To Kaiwen

PREFACE

Dynamics is a subject dealing with objects (natural or artificial) with mass in motion. It regards two fundamental questions.

1) How do objects move?
2) Why do objects move?

The first question leads to the study of *Kinematics* and the second one to *Kinetics*. In *Kinematics*, motion of an object is described and quantified by its time-varying displacement (linear: $d = d(\text{t})$ / angular: $\theta = \theta(\text{t})$), velocity (linear: $v = v(\text{t})$ / angular: $\omega = \omega(\text{t})$), and acceleration ($a=a(t)$). These dynamic parameters (d, v, a) carry different forms in various coordinate systems. In *Kinetics*, the relationship between dynamic parameters and mechanical parameters (force F, moment M, work W, energy U, impulses, linear momentum L, angular momentum H) are investigated in order to explain the motion of an object.

Students intending to succeed in *Dynamics* must be equipped with college-level *Calculus* and *Statics*. Topics like coordinate systems, trigonometry, vector analysis, differentiation, and integration are usually covered in the modern approach of college-level *Calculus*. In *Statics*, free-body diagram, force equilibrium, and moment equilibrium are typically included.

Over the past several years, I have been teaching *Dynamics* at the UMass Lowell to undergraduate students in Civil and Environmental Engineering, Mechanical Engineering, and Chemical Engineering. While there is a trend to use many illustrated examples to introduce the topics in *Dynamics*, many times I found students having difficulties grasping the big picture of *Dynamics*. Rather, many students treat the topics in *Dynamics* as disconnected dots, resulting in tedious efforts to learn these topics by memorizing specialized equations. I hope this book can solve this problem by presenting a systematic perspective on all topics in *Dynamics*.

It is this author's intention to keep this manuscript as concise as possible. There are already many comprehensive titles on *Dynamics*, typically presented with sufficient illustrations to exemplify concepts in *Dynamics*. It is also this author's intention to use abstracted sketches to describe concepts in *Dynamics*. Analytical reasoning can be fostered by identified key information and, in many cases, is better presented by simplified illustrations.

As always, curiosity is the key to deepen one's understanding on any subject. Students will find themselves at an elevated level of understanding if they care to constantly ask "why?" on everything they learn. This suggested approach is certainly time consuming, but it is also the only way to mastering any subject. The view on the top of a mountain is always magnificent, but only those who are willing to climb up can enjoy it.

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December 2020

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FUNDAMENTALS

Before introducing the topics in *Dynamics*, it is important to make sure that students are equipped with necessary knowledge from *Calculus* and *Statics*. The mission of this chapter is to help students review related topics in *Calculus* and *Statics* before they take on the journey in *Dynamics*. Related topics like coordinate systems (rectangular/Cartesian, polar, cylindrical, spherical) and the conversions among them, space-time conversion, and mass moment of inertia are reviewed in this chapter.

1.1 Fundamental Axiom and Assumptions

Before we start the journey, we must be aware of all fundamental axiom and assumptions and their consequences in *Dynamics* in order to understand the limitations of derived formulae in *Dynamics*. The fundamental axiom in *Dynamics* is

> All objects considered in *Dynamics* must be made of different kinds of matter with mass. The consequence of considering mass in *Dynamics* is the inertia (linear and/or angular) caused by changing the dynamic state of objects.

Two fundamental assumptions are

- **Rigidity assumption** – All objects considered in *Dynamics* are rigid body, suggesting that they are undeformable (incompressible, inextensible) when in motion. The consequence is that all mechanical effects (e.g., stress/strain distribution) inside an object are not considered in *Dynamics*.
- **Velocity assumption** – All velocities considered in *Dynamics* are much less than the speed of light (2.99792458×10^8 meters/second). The consequence is that relative motion holds true in *Dynamics*.

1.2 Coordinate Systems

To answer the first basic question in *Dynamics*, "How do objects move?" one must be able to describe a motion. Using coordinates in selected coordinate systems serves the purpose. For the problems considered in *Dynamics*, the following coordinate systems are used.

- The rectangular/Cartesian coordinate system
- The polar coordinate system
- The cylindrical coordinate system
- The spherical coordinate system

In **rectangular/Cartesian coordinate** systems, one-dimensional (1-D), two-dimensional (2-D), and three-dimensional (3-D) coordinates are available for dynamic parameters (displacement, velocity, and acceleration) of different dimensionalities. Since these parameters are directional, it is necessary to use vector representation to define them. Vectors are 1-D tensors, while scalars are zero-dimensional (0-D) tensors and matrices 2-D tensors.

The axis vectors in rectangular coordinate systems are denoted by their unit vectors $(\vec{x}, \vec{y}, \vec{z})$. It is also customary to use $(\vec{\imath}, \vec{\jmath}, \vec{k})$ as the unit vectors in rectangular coordinate systems.

In addition, $\vec{\imath} \perp \vec{\jmath}$, $\vec{\jmath} \perp \vec{k}$, $\vec{k} \perp \vec{\imath}$, and $|\vec{\imath}| = |\vec{\jmath}| = |\vec{k}| = 1$. Rectangular coordinates are denoted by (x, y, z) with axes defined by (X, Y, Z).

$$\vec{s} = s_x\vec{x} + s_y\vec{y} + s_z\vec{z} \tag{1-1}$$

where s_x, s_y, s_z = linear components of vector $\vec{s}$ on X, Y, Z, respectively. The magnitude of $\vec{s}$ is determined by $s = |\vec{s}| = \sqrt{{s_x}^2 + {s_y}^2 + {s_z}^2}$. Fig. 1-1 shows vector $\vec{s}$ in a 3-D rectangular coordinate system.

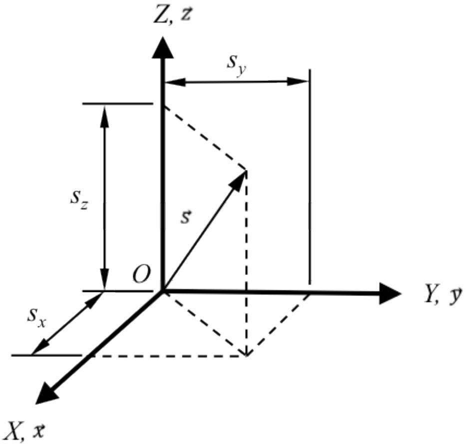


Fig. 1-1. 3-D rectangular/Cartesian coordinate representation of vector $\vec{s}$

In **polar coordinate** systems, 1-D and 2-D coordinates are available. Polar coordinates are denoted by (r, Θ) with axes defined by (R, Θ) and unit vectors by $(\vec{r}, \vec{\theta})$. The polar coordinate representation of vector $\vec{s}$ is

$$\vec{s} = s_r\vec{r} + s_\theta\vec{\theta} \quad (1\text{-}2)$$

where s_r, s_θ = linear and angular components of vector $\vec{s}$ on R, Θ, respectively. The magnitude of $\vec{s}$ is $s = s_r$. Fig. 1-2 shows vector $\vec{s}$ in a 2-D polar coordinate system.

In **cylindrical coordinate** systems, 1-D, 2-D, and 3-D coordinates are available. Cylindrical coordinate systems are denoted by (r, Θ, z) with axes defined by (R, Θ, Z) and unit vectors by $(\vec{r}, \vec{\theta}, \vec{z})$. The cylindrical coordinate representation of vector $\vec{s}$ is

$$\vec{s} = s_r\vec{r} + s_\theta\vec{\theta} + s_z\vec{z} \quad (1\text{-}3)$$

where s_r, s_θ, s_z = linear, angular, and linear components of vector $\vec{s}$ on R, Θ, Z, respectively. The magnitude of $\vec{s}$ is $s = \sqrt{{s_r}^2 + {s_z}^2}$. Fig. 1-3 shows vector $\vec{s}$ in a 3-D cylindrical coordinate system.

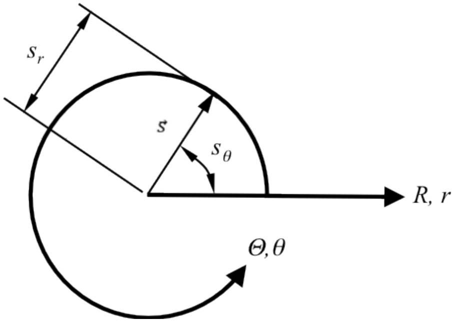


Fig. 1-2. 2-D polar coordinate representation of vector $\vec{s}$

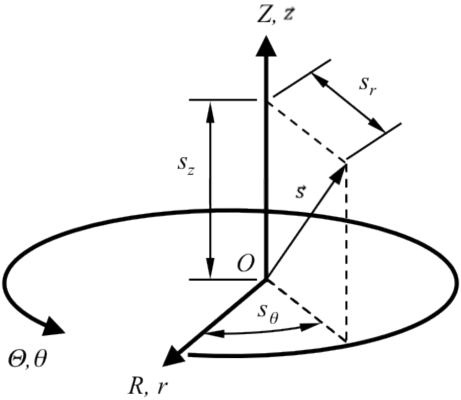


Fig. 1-3. 3-D cylindrical coordinate representation of vector $\vec{s}$

In **spherical coordinate** systems, 1-D, 2-D, and 3-D coordinates are available. Spherical coordinate systems are denoted by (r, θ, ϕ) with axes defined by (R, Θ, Φ) and unit vectors by $(\vec{r}, \vec{\theta}, \vec{\phi})$. The cylindrical coordinate representation of vector $\vec{s}$ is

$$\vec{s} = s_r\vec{r} + s_\theta\vec{\theta} + s_\phi\vec{\phi} \tag{1-4}$$

where s_r, s_θ, s_ϕ = linear, angular, and angular components of vector $\vec{s}$ on R, Θ, Φ, respectively. The magnitude of $\vec{s}$ is $s = s_r$. Fig. 1-4 shows vector $\vec{s}$ in a 3-D spherical coordinate system.

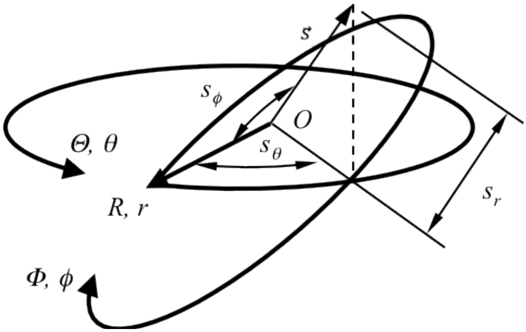


Fig. 1-4. 3-D spherical coordinate representation of vector $\vec{s}$

Conversion among coordinate systems is necessary in *Dynamics* when studying the interaction between two objects. From rectangular to cylindrical coordinates,

$$s_r = \sqrt{{s_x}^2 + {s_y}^2} \tag{1-5}$$
$$s_\theta = \tan^{-1}\left(\frac{s_y}{s_x}\right) \tag{1-6}$$
$$s_z = s_z \tag{1-7}$$

From cylindrical to rectangular coordinates,

$$s_x = s_r \cos(s_\theta) \tag{1-8}$$
$$s_y = s_r \sin(s_\theta) \tag{1-9}$$
$$s_z = s_z \tag{1-10}$$

From rectangular to spherical coordinates,

$$s_r = \sqrt{{s_x}^2 + {s_y}^2 + {s_z}^2} \tag{1-11}$$
$$s_\theta = \tan^{-1}\frac{s_y}{s_x} \tag{1-12}$$
$$s_\phi = \tan^{-1}\frac{\sqrt{{s_x}^2+{s_y}^2}}{s_z} \tag{1-13}$$

From spherical to rectangular coordinates,

$$s_x = s_r \cos(s_\theta) \sin(s_\phi) \tag{1-14}$$

$$s_y = s_r \sin(s_\theta) \sin(s_\phi) \tag{1-15}$$
$$s_z = s_r \cos(s_\phi) \tag{1-16}$$

Conversion between polar and rectangular coordinates is a special case of the one between cylindrical and rectangular coordinates, which can be obtained from Eqs. (1-5) ~ (1-10).

1.3 Vector Algebra

Arithmetic operations between two vectors fall into the discipline of vector algebra or linear algebra. In *Dynamics*, only simply arithmetic operations are used, including addition, subtraction, and multiplication. Their physical meanings in *Dynamics* are also introduced in this section. However, before we discuss these operations, properties of a vector must be introduced.

Magnitude/amplitude – The magnitude or amplitude of a vector is defined by the summation of all components ($s = |\vec{s}|$). Since a vector can have various representations in different coordinate systems, its magnitude is determined differently, depending on the coordinate system. These representations are listed in the following.

- Rectangular/Cartesian coordinates –
$$s = \sqrt{{s_x}^2 + {s_y}^2 + {s_z}^2} \tag{1-17}$$

- Polar coordinates –
$$s = s_r \tag{1-18}$$

- Cylindrical coordinates –
$$s = \sqrt{{s_r}^2 + {s_z}^2} \tag{1-19}$$

- Spherical coordinates –
$$s = s_r \tag{1-20}$$

Direction – The direction of a vector is defined by its directional vector. A directional vector is the unit vector of any arbitrary vector, which has a magnitude of unity. Depending on the coordinate system, a directional vector takes the following forms.

- Rectangular/Cartesian coordinates –

$$\hat{s} = \frac{\vec{s}}{\sqrt{{s_x}^2+{s_y}^2+{s_z}^2}} = \frac{s_x}{s}\vec{x} + \frac{s_y}{s}\vec{y} + \frac{s_z}{s}\vec{z} \tag{1-21}$$

- Polar coordinates –

$$\hat{s} = \frac{\vec{s}}{s} = \frac{s_r}{s}\vec{r} + \ s_\theta\vec{\theta} \tag{1-22}$$

- Cylindrical coordinates –

$$\hat{s} = \frac{\vec{s}}{\sqrt{{s_r}^2+{s_z}^2}} = \frac{s_r}{s}\vec{r} + \ s_\theta\vec{\theta} + \ \frac{s_z}{s}\vec{z} \tag{1-23}$$

- Spherical coordinates –

$$\hat{s} = \frac{\vec{s}}{s} = \frac{s_r}{s}\vec{r} + \ s_\theta\vec{\theta} + \ s_\phi\vec{\phi} \tag{1-24}$$

Addition/subtraction – Addition/subtraction between two vectors is carried out at the component level. Its result is another vector. Consider two vectors $\vec{s}$ and $\vec{d}$ in Cartesian coordinates.

$$\vec{s} \pm \vec{d} = (s_x \pm d_x)\vec{x} + \left(s_y \pm d_y\right)\vec{y} + (s_z \pm d_z)\vec{z} \tag{1-25}$$

In *Dynamics*, addition/subtraction between two vectors leads to a resultant vector (e.g., displacement, velocity, or acceleration). This operation is frequently encountered in relative motion. Furthermore, subtraction between two position vectors defines a displacement vector.

Multiplication – There are two major multiplication operations used in *Dynamics*, scalar or dot product and vector or cross product. The **scalar/dot product** between two vectors is a scalar, which is defined by

$$\vec{s} \cdot \vec{d} = s_x d_x + s_y d_y + s_z d_z \tag{1-26}$$

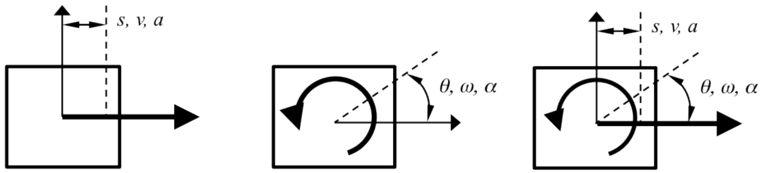
The **vector/cross product** between two vectors results in a vector, which is defined by

$$\vec{s} \times \vec{d} = \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ s_x & s_y & s_z \\ d_x & d_y & d_z \end{vmatrix} \tag{1-26}$$

$$= \left(s_y d_z - s_z d_y\right)\vec{x} + (s_z d_x - s_x d_z)\vec{y} + \left(s_x d_y - s_y d_z\right)\vec{z} \tag{1-27}$$

1.4 Dynamic Parameters and Space-Time Conversion

In *Dynamics*, information regarding the motion of an object can be expressed in the time domain or the space domain. Physically, an object can move in one of the three modes: 1) translational/linear motion, 2) rotational/angular motion, and 3) general motion (translation and rotation), as shown in Fig. 1-5. These modes are further elaborated in the following.



(a) Translational/linear motion (b) Rotational/angular motion (c) General motion

Fig. 1-5. Three fundamental modes in *Dynamics*

1) **Translational/linear motion** – In translational motion, three **linear dynamic parameters** are used in describing the motion of an object; namely **linear displacement *s***, **linear velocity *v*,** and **linear acceleration *a***. In the vector form, relationships among these parameters are defined by

$$\vec{s}(t) = \vec{S}(t_1) - \vec{S}(t_0) = \vec{S}_1 - \vec{S}_0 \tag{1-28}$$

$$\vec{v}(t) = \frac{d\vec{s}(t)}{dt} \tag{1-29}$$

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} \tag{1-30}$$

where $\vec{S}_1$ = position vector at time t_1 and $\vec{S}_0$ = position vector at time t_0. Eqs. (1-29) and (1-30) both provide the expression of *dt* and, thus, can be equated, leading to

$$\frac{d\vec{s}(t)}{\vec{v}(t)} = \frac{d\vec{v}(t)}{\vec{a}(t)} \tag{1-31}$$

or simply

$$\vec{a}d\vec{s} = \vec{v}d\vec{v} \tag{1-32}$$

which is valid regardless of any given space or time condition.

By definition, displacement is the change of position between two time instants (in this case, t_0 and t_1). Velocity is the instantaneous change of displacement between two time instants with infinitesimal difference in time. Acceleration is the instantaneous change of velocity between time instants with infinitesimal difference in time. This suggests that

$$\epsilon = t_1 - t_0 \tag{1-33}$$

where ϵ = infinitesimal difference in time. Should the difference between two time instants become significant, the average version of linear dynamic parameters is used.

$$\vec{s}_{\text{avg}}(\Delta t) = \vec{S}_1 - \vec{S}_0 \tag{1-34}$$
$$\vec{v}_{\text{avg}}(\Delta t) = \frac{\Delta\vec{s}(\Delta t)}{\Delta t} \tag{1-35}$$
$$\vec{a}_{\text{avg}}(\Delta t) = \frac{\Delta\vec{v}(\Delta t)}{\Delta t} \tag{1-36}$$

where Δt = time difference and $\Delta t \gg \epsilon$. $\vec{s}_{\text{avg}}$ is the average linear displacement, $\vec{v}_{\text{avg}}$ the average linear velocity, and $\vec{a}_{\text{avg}}$ the average linear acceleration. Eqs. (1-28) to (1-30) and Eqs. (1-34) to (1-36) are all in the time domain.

In some cases, it is useful to express these parameters in the space domain. To do so, the conversion between space and time variables is needed. Three examples are provided to demonstrate such conversion in linear motion.

I. Constant displacement (static problem)

$$s = c_1 \tag{1-37}$$
$$v = 0 \tag{1-38}$$
$$a = 0 \tag{1-39}$$

where c_1 = constant. In static problems, no space-time conversion is possible.

II. Constant velocity

$$s(t) = c_1 + c_2 t \tag{1-40}$$
$$v = c_2 \tag{1-41}$$

$$a = 0 \tag{1-42}$$

where c_2 = constant. Eq. (1-40) leads to

$$t(s) = \frac{1}{c_2}(s - c_1) \tag{1-43}$$

III. Constant acceleration

$$s(t) = c_1 + c_2 t + c_3 t^2 \tag{1-44}$$
$$v(t) = c_2 + 2c_3 t \tag{1-45}$$
$$a = 2c_3 \tag{1-46}$$

where c_3 = constant. Applying the quadratic formula to Eq. (1-44) leads to

$$t(s) = \frac{1}{2c_3}\left[\sqrt{c_2^2 - 4c_3(c_1 - s)} - c_2\right] \tag{1-47}$$

Furthermore,

$$v(s) = \sqrt{c_2^2 - 4c_3(c_1 - s)} \tag{1-48}$$

Example 1-1: Constant acceleration in a gravitational field.

In a gravitational field, motion of objects is in constant acceleration such that $a = a_g$ = gravitational acceleration. Therefore, from Eq. (1-44), $a_g = 2c_3$ or $c_3 = \frac{1}{2}a_g$. Equation (1-45) then becomes

$$t(s) = \frac{1}{a_g}\left[\sqrt{c_2^2 - 2a_g(c_1 - s)} - c_2\right] \tag{E1-1}$$

As for Eq. (1-43), the dimension of c_2 must be same as $v(t)$, suggesting that $c_2 = v_0$.

$$v(t) = v_0 + a_g t \tag{E1-2}$$

Substituting $c_2 = v_0$ and $c_3 = \frac{1}{2}a_g$ into Eq. (1-42), we have

$$s(t) = c_1 + v_0 t + \frac{1}{2}a_g t^2 \tag{E1-3}$$

Dimensional consistency demands the unit of c_1 to be same with $s(t)$, suggesting that $c_1 = v_0$. Finally,

$$s(t) = s_0 + v_0 t + \frac{1}{2} a_g t^2 \tag{E1-4}$$

Eqs. (E1-2) and (E1-4) can also be derived by integrating Eqs. (1-44) and (1-43), respectively.

2) **Rotational/angular motion** – In angular motion, three **angular dynamic parameters** are used in describing the motion of an object; namely **angular displacement $\boldsymbol{\theta}$, angular velocity $\boldsymbol{\omega}$**, and **angular acceleration $\boldsymbol{\alpha}$**. In the vector form, relationships among these parameters are defined by

$$\vec{\theta}(t) = \vec{\theta}(t_1) - \vec{\theta}(t_0) = \vec{\theta}_1 - \vec{\theta}_0 \tag{1-49}$$

$$\vec{\omega}(t) = \frac{d\vec{\theta}(t)}{dt} \tag{1-50}$$

$$\vec{\alpha}(t) = \frac{d\vec{\omega}(t)}{dt} \tag{1-51}$$

where $\vec{\theta}_1$ = angular position vector at time t_1 and $\vec{\theta}_0$ = angular position vector at time t_0. Eqs. (1-50) and (1-51) both provide the expression of *dt* and can be equated, leading to

$$\frac{d\vec{\theta}(t)}{\vec{\omega}(t)} = \frac{d\vec{\omega}(t)}{\vec{\alpha}(t)} \tag{1-52}$$

or simply

$$\vec{\alpha} d\vec{\theta} = \vec{\omega} d\vec{\omega} \tag{1-53}$$

which is valid regardless of any given space or time condition.

By definition, angular displacement is the angular change of position between two time instants (in this case, t_0 and t_1). Angular velocity is the instantaneous change of angular displacement between two time instants with infinitesimal difference in time. Angular acceleration is the instantaneous change of angular velocity between time instants with infinitesimal difference in time. Similar discussion in Eqs. (1-33)~(1-36) for linear motion applies to angular motion.

The space-time conversion in linear motion can also be carried out in angular motion. Three examples are provided to demonstrate such conversion in angular motion.

I. Constant displacement (static problem)

$$\theta = c_1 \tag{1-54}$$
$$\omega = 0 \tag{1-55}$$
$$\alpha = 0 \tag{1-56}$$

where c_1 = constant. As expected, no space-time conversion is possible in static problems.

II. Constant velocity

$$\theta(t) = c_1 + c_2 t \tag{1-57}$$
$$\omega = c_2 \tag{1-58}$$
$$\alpha = 0 \tag{1-59}$$

where c_2 = constant. Eq. (1-57) leads to

$$t(\theta) = \frac{1}{c_2}(\theta - c_1) \tag{1-60}$$

III. Constant acceleration

$$\theta(t) = c_1 + c_2 t + c_3 t^2 \tag{1-61}$$
$$\omega(t) = c_2 + 2c_3 t \tag{1-62}$$
$$\alpha = 2c_3 \tag{1-63}$$

where c_3 = constant. Applying the quadratic formula to Eq. (1-61) leads to

$$t(\theta) = \frac{1}{2c_3}\left[\sqrt{c_2^2 - 4c_3(c_1 - \theta)} - c_2\right] \tag{1-64}$$

Furthermore,

$$\omega(s) = \sqrt{c_2^2 - 4c_3(c_1 - \theta)} \tag{1-65}$$

3) **General motion (translation and rotation)** – In general motion, an object carries both linear and angular components of dynamic parameters. The result of this superposition is the difference between the linear dynamic parameters at the center of rotation and everywhere else. In general motion, we have the following six equations to use.

$$\vec{v}(t) = \frac{d\vec{s}(t)}{dt} \tag{1-29}$$

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} \tag{1-30}$$

$$\vec{a}d\vec{s} = \vec{v}d\vec{v} \tag{1-32}$$

$$\vec{\omega}(t) = \frac{d\vec{\theta}(t)}{dt} \tag{1-50}$$

$$\vec{\alpha}(t) = \frac{d\vec{\omega}(t)}{dt} \tag{1-51}$$

$$\vec{\alpha}d\vec{\theta} = \vec{\omega}d\vec{\omega} \tag{1-53}$$

Superposition of linear and angular components at the center of rotation applies because there is no linear motion at the center of rotation. The general motion at the locations other than the center of rotation will be discussed in **Chapter 7 Two-dimensional General Motion**.

1.5 Mass Moment of Inertia

Recall Newton's Second Law of Motion.

"Force is equal to the change in momentum per change in time. For a constant mass, force equals mass times acceleration."

—— Issac Newton (1686),
"Principia Mathematica Philosophiae Naturalis"

When moving from rest in linear motion, an object needs to overcome its linear inertia (mass m) by introducing a force F.

$$F = ma \tag{1-51}$$

where a = linear acceleration (m/s^2 or ft/s^2). To determine how much force is needed to achieve a given acceleration, the mass must be known.

When moving from rest in angular motion, an object needs to overcome its angular inertia (mass moment of inertia, denoted by *I*) by introducing a force moment *M*.

$$M = I\alpha \qquad (1\text{-}52)$$

where α = angular acceleration (rad./s^2). To determine how much moment is needed to achieve a given angular acceleration, the mass moment of inertia must be known. Mass moment of inertia carries the unit of kg-m/s^2 in the SI Unit and slug-ft/s^2 in the FPS system.

The value of mass moment of inertia of an object is calculated by the following two properties.

1) Reference point – It indicates the centre of rotation; with respect to (w.r.t.) a specific point.
2) Axis of rotation – It indicates the direction of rotation; w.r.t. a specific axis.

As a result, mass moment of inertia should be denoted by $(I_i)_j$ where subscript i denotes a reference point and subscript j the axis of rotation. Therefore, an object has many values of mass moment of inertia, depending on the selected reference point and the selected axis of rotation. Consequently, a complete description of mass moment of inertia is, for example, I_x^0 or $(I_0)_x$ where *0* denotes a reference point and *x* an axis of rotation.

There are two approaches to calculate *I*.

1) Integration – By definition, mass moment of inertia is found by the general form.

$$I = \int_V r^2 dm \qquad (1\text{-}53)$$

where *V* = volume of the object, *r* = distance variable perpendicular to the axis of rotation, dm = derivative of mass. For instance, *y* and *z* can be used to find $(I_0)_x$ and *x* and *z* to find $(I_0)_y$, depending on the mathematical ease of integral.

2) Parallel axis theorem – The mass moment of inertia w.r.t. any reference point other than the centre of mass (or centre of gravity if uniform density) can be found by

$$(I_1)_i = (I_0)_i + md^2 \tag{1-54}$$

where $(I_1)_i$ = mass moment of inertia w.r.t. the i-axis passing through point 1, $(I_0)_i$ = mass moment of inertia w.r.t. the i-axis passing through the centre of mass (point 0 in this case), m = mass, and d = distance between two **parallel** axes. i could be x or y or z in a Cartesian coordinate system.

Since $d^2 \geq 0$ and $m \geq 0$, it is evident that the mass moment of inertia at the centre of mass must be the minimum of all possible values of $(I_1)_i$ w.r.t. any point else. Meanwhile, since the i-axis passing through point 1 must be in parallel with the i-axis passing through the centre of mass (they are both i-axis), hence the name *parallel axis theorem*.

Eq. (1-54) can also be written as

$$(I_O)_i = (I_G)_i + md^2 \tag{1-55}$$

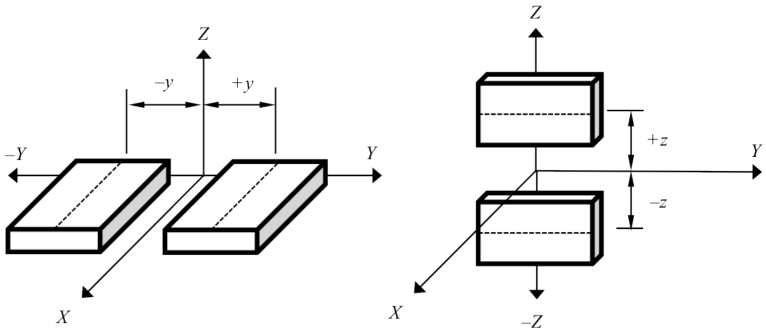
where $(I_O)_i$ = mass moment of inertia w.r.t. the i-axis passing through point O (center of rotation), $(I_G)_i$ = mass moment of inertia w.r.t. the i-axis passing through the center of mass. Both Eq. (1-54) and (1-55) represent the mathematical form of **parallel axis theorem**.

The following principles provide instrumental values to calculating mass moment of inertia.

- Principle of symmetry – From the definition of mass moment of inertia in Eq. (1-53), positive or negative sign of variable r does not make any difference in the value of *I*. This leads to the principle of symmetry in the calculation of mass moment of inertia.

$$\int_V (+r)^2 dm = \int_V (-r)^2 dm \tag{1-56}$$

Fig. 1-6 provides examples of symmetric objects (with identical mass moment of inertia) w.r.t. to *X* and *Y* axes in a Cartesian coordinate system.



(a) Symmetry w.r.t. Z axis (b) Symmetry w.r.t. Y axis

Fig. 1-6. Objects with identical mass moment of inertia due to symmetry

- Principle of equivalence – For some geometries, they have identical mass moment of inertia if change of shapes does not change the center of mass. In Fig. 1-7, all three triangles have same mass moment of inertia w.r.t y axis.

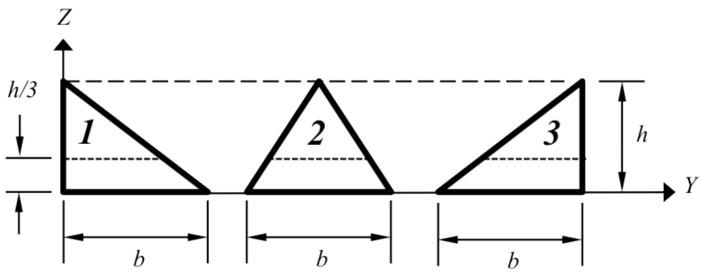


Fig. 1-7. Objects with identical mass moment of inertia due to equivalence

- Principle of superposition – By utilizing parallel axis theorem, an object can be considered as an assembly of small objects of regular shape. This is written by

$$(I_O)_i = \sum_{k=1}^{n}\left[(I_G)_i^k + md_k^2\right] \tag{1-57}$$

where $(I_G)_i^k$ = mass moment of inertia w.r.t. the i-axis passing through the center of mass of the k^{th} small object, d_k = parallel distance from the mass center of the k^{th} small object to the axis of rotation. For example, the leftmost triangle in Fig. 1-8 can be considered as the assembly of small objects 1 and 2.

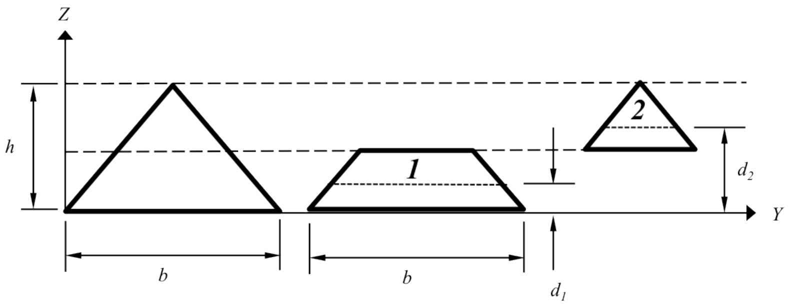


Fig. 1-8. Superposition of objects

One last note on mass moment of inertia is the significance of **density** ρ. Since mass is the product of density ρ and volume V, Eq. (1-53) can be written as

$$I = \int_V r^2(\rho dV) \tag{1-58}$$

Should we assume homogeneous objects, density is constant.

$$I = \rho \int_V r^2 dV \tag{1-59}$$

However, most engineering structures are not homogeneous and do not have constant density. Therefore, Eq. (1-58) must re-written into the following for composite sections and structures.

$$I = \sum_{i=1}^{n} \int_V r^2(\rho_i dV_i) \tag{1-60}$$

where n = total number of materials, ρ_i = density of the i^{th} material, and dV_i = unit volume of the i^{th} material. Furthermore, also from Eq. (1-58), a small object may possess greater values of mass moment of inertia than a large object, if the small object is made of denser/heavier material.

Example 1-2: Mass moment of inertia of a triangular plate in various orientations.

To better understand the concept and calculation of mass moment of inertia, we consider three orientations of a triangular plate in this example.

(a) Orientation I
Consider orientation I as shown in Fig. E1-1. The thickness of the plate is w, and the density is ρ. Determine I_y and I_z.

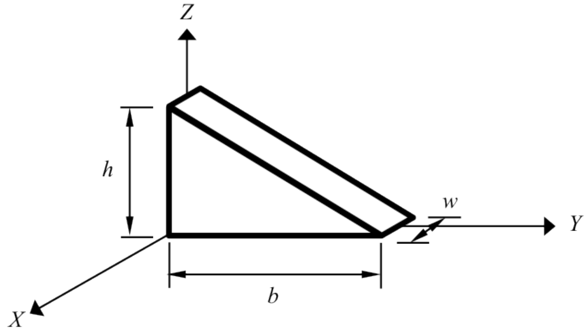
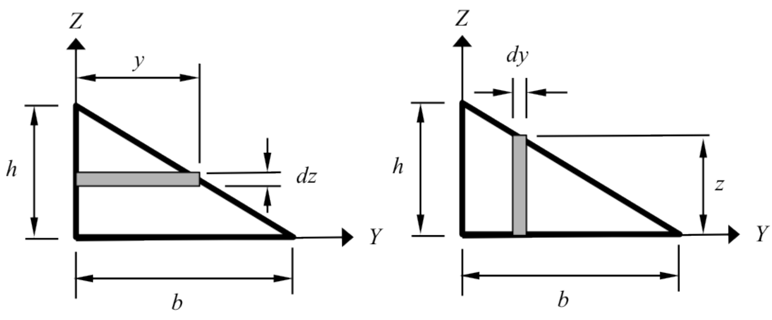


Fig. E1-1. Orientation I of a triangular plate

$$I_y = \int_V z^2 dm \tag{E1-5}$$

$$I_z = \int_V y^2 dm \tag{E1-6}$$

Consider the integration scheme in Fig. E1-2 (a) for I_y and Fig. E1-2 (b) for I_z.



(a) Integration scheme for I_y (b) Integration scheme for I_z