The Direct Integration Method for Elastic Analysis of Nonhomogeneous Solids

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By

Yuriy Tokovyy and Chien-Ching Ma

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To the brilliant memory of Prof. Vasyl' M. Vihak (Vigak), 1936 – 2003,
and Prof. Viatcheslav (Slava) V. Meleshko, 1951 – 2011,
to whom we are greatly indebted for the contribution
into our scientific endeavors

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PREFACE

The recent progress in computational methods and computer-based simulation models contributed significantly to the efficiency of scientific research in different areas. In the area of thermomechanics, for example, the widespread implementation of the modern computer methods is motivated by the complexity of scientific problems addressing the thermo-mechanical phenomena in material structures, the coupling of processes and physicochemical fields, variety in shape and material properties of the structures or their assemblies, multi-field nature of impacts, time and cost constraints for the computation and analysis, etc. Such a complexity level makes it nearly impossible to derive a solution analytically. This reason tips the scale of interest for students, scholars, and research-engineers towards the "fast and efficient" numerical methods over the analytical ones, which may seem to be "potentially insufficient" and "too sophisticated". This also widens the gap between the "modern" and "pre-computer" scientists in their approach. So maybe it is the time for analytical methods to take their place in the "museum of science" alongside the slide rule and arithmometer?

The authors sincerely believe that the depreciation of analytical methods, in many cases, fails to capture important features of the studied processes and phenomena. In our experience, the implementation solely of even the newest and most efficient numerical methods for the analysis of structures of specific geometries (for example, propellant tanks of launch vehicles) is related to certain complications. One of those is a very large number of equations to solve, which presents a challenge even for modern computational facilities. Therefore, the original boundary value problem on the mechanical analysis of the entire structure is to be segmented into a number of sub-problems formulated for the representative elements of the structure. This, however, imposes a new problem on the adequate evaluation of the boundary conditions for the considered structural element allowing for the simulation of the impact caused by the remaining parts of the structure. The efficiency in solving the latter problem depends on understanding the critical stress behavior which calls for implementing analytical procedures for a clear cause-consequence analysis.

This makes it clear that the progress in the analysis of complex mechanical problems depends, in the final count, on the synergy of analytical and numerical methods. In this regard, we can refer to the motto

of the book [101] by R. W. Hamming: "*The purpose of computing is insight, not numbers*". From this viewpoint, the main demand for the analytical methods is not just constructing a solution (which is now not the final goal of the research but rather an intermediate step), but presenting it in the most convenient form that can be used for further analysis. Such a solution form in mechanics of solids implies the explicit analytical relationship between the loadings and stress-strain fields.

In this book, we present an analytical method for the thermoelastic analysis inhomogeneous solids, the central idea of which is representing the stresses in the form of explicit analytical dependencies on the mechanical and thermal loadings. This method is based on the concept of direct integration of the governing equations of the elasticity and thermoelasticity problems from the "first principles", i.e., by operating with the stresses or displacements without implementation of the potential functions of higher differential order. The method is oriented towards the integrodifferential relationship between the stress-tensor components, which is derived on the basis of equilibrium equations in terms of stresses and thus is irrespective of the material properties. This fact makes this method very attractive for solving thermoelasticity problems in anisotropic and inhomogeneous solids.

The hypotheses and models underlying classical theories of elasticity and thermoelasticity mostly assume the elastic properties of isotropic and anisotropic materials to be constant. However, later, primarily due to a deeper empirical study of the elastic behavior of real-life structures and the needs of engineering practice, it became necessary to take into account the dependence of the material moduli on spatial coordinates. Materials with such properties are known as inhomogeneous materials. The studies of the thermomechanical behavior of inhomogeneous structures under the action of force and thermal loadings attract the attention of specialists in both academia and industry. In particular, this is due to the development of the concept of functional-gradient materials (FGM) and the latest technologies for the formation of inhomogeneous structures with a predetermined distribution profiles of thermophysical and mechanical properties, which are intensively introduced and studied in scientific and industrial centers around the world.

The development of methods for the analysis of the thermomechanical behavior of inhomogeneous or FGM solids is concerned with significant difficulties, caused primarily by the need to solve governing differential equations with variable coefficients within the framework of corresponding problems of thermomechanics. The classical methods, in the vast majority, fail to meet urgent needs in this field, the main of which is to determine the optimal distribution of material characteristics to ensure certain functional

parameters of a structure as a whole, as well as optimal control of their thermostressed state. From this point of view, the determination of the stress-strain state is not the final goal of the research, but only an intermediate stage, which should provide an analytical solution of the direct problem, which satisfies the boundary and initial conditions, fundamental principles and modeling constraints of solids mechanics, and is in the form of an explicit functional dependence on loadings and material properties, which is critical for further application.

This book generalizes the results in solving two-dimensional elasticity and thermoelasticity problems for anisotropic inhomogeneous solids we generated over two decades by developing the direct integration method. We attempted to emphasize the advantages and, what is more important, the disadvantages of the method for solving practical problems.

The book consists of four chapters. The first chapter is devoted to the historical aspects of the research into the thermomechanical performance of inhomogeneous solids along with the review of the dominant analytical methods in this subject area. We intentionally emphasized the earlier studies over the new ones for the reason that, unfortunately, many "new" results published in recent decades reproduce (completely or in some part) the older ones, which we may explain by the lack of knowledge about the studies provided in pioneering research works. The second chapter presents the application of the direct integration method for solving thermoelasticity problems for orthotropic inhomogeneous infinite, semi-infinite, and finite solids in the Cartesian coordinate system. The third chapter presents the analysis of orthotropic inhomogeneous annuli in the polar coordinates. The fourth chapter deals with the construction of solutions to thermoelasticity problems for infinite, semi-infinite, and finite solids in the cylindrical coordinate system. Under the assumption that the material properties are arbitrary functions of the spatial coordinates, besides the construction of solutions in the form of explicit dependencies on the force and thermal loadings, we provided a technique of deriving the necessary existing conditions for the stresses and displacements in terms of the loadings applied, and the one-to-one relationship between the stresses and displacements on the boundary of the considered solids. In every considered coordinate system, we discussed the cases of the material inhomogeneity profiles, which allow for comparatively simple analytical solutions of thermoelasticity problems. We hope, these results may be of interest for scientists and engineers working in the area of thermal stresses with the focus on the effects caused by the material anisotropy and inhomogeneity, as well as for university students with the specialty in mechanical and civil engineering and methods mathematical physics.

We are indebted to many people for this book. First of all, the book is dedicated to the brilliant memory of Professor Vasyl Vihak (Vigak, if the name is transliterated from Russian), who initiated the research into the direct integration of the thermoelasticity equations, and Professor Viatcheslav (Slava) Meleshko, whose valuable comments and suggestions helped us with clarifying a number of critical points in our research. We are also thankful to the researchers of the Departments of Solid Mechanics and Thermomechanics at the Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of the National Academy of Sciences of Ukraine (Lviv, Ukraine) and the Fracture Mechanics Laboratory at the Department of Mechanical Engineering, Colledge of Engineering of the National Taiwan University (Taipei, Taiwan) for the discussions, important critics and valuable comments on different aspects of the presented material. The authors would like to thank the Ministry of Science and Technology of the Republic of China for financially supporting the research projects which provide valuable results in this book.

Lviv – Taipei
Yuriy Tokovyy
Chien-Ching Ma
August 2020

CHAPTER ONE

ELASTIC AND THERMOELASTIC ANALYSIS OF INHOMOGENEOUS SOLIDS

1.1. Introduction

In the first part of the nineteenth century, the classical theory of elasticity broke off into a separate academic discipline within the continua mechanics owing to the fundamental contributions of scientists, such as Claude Louis Marie Henri Navier (1785–1836), Augustin Louis Cauchy (1789–1857), and Siméon Denis Poisson (1781–1840), and others [34, 281, 284]. This theory lays the foundation by which to analyze the elastic response of solids to static and dynamic force loadings, which is crucial to projecting and evaluating the mechanical performance of the structural elements of buildings and mechanisms. The central hypotheses of this theory are based on the assumption that materials are continuous (i.e., each material point of a considered solid can be bijectively represented by a point in three-dimensional Euclidean space) and homogeneous (i.e., mechanical properties are the same at any point in a solid).

By the end of the nineteenth century however, it became obvious that the assumption of material homogeneity is unable to capture several important features pertaining to the mechanical performance of solids. A lack of homogeneity is associated with structural imperfections or microdefects, the consolidation of which within a macrovolume can cause profound disturbances of mechanical and thermal fields. Materials with such properties are known as *inhomogeneous, nonhomogeneous,* or *heterogeneous* [169, 202]. Since that time, there has been considerable research on the analysis of inhomogeneous solids with concern to various aspects and applications [32, 35, 36, 116, 142]. In recent years, this trend has accelerated due to the widespread implementation of modern materials with advanced properties. There has been a particular focus on functionally-graded materials (FGM), the technology of which allows for the intentional modification of material-variation profiles during fabrication [265, 267] in order to meet specific thermo-mechanical

performance requirements. This has led to a proliferation of scientific publications on the various aspects of modeling and analysis of inhomogeneous solids. Sadly, many of these papers reiterate results published in older articles.

Maxwell [170, pp. xiii–xiv] pointed out that "*it is of great advantage to the student of any subject to read the original memoirs on that subject, for science is always most completely assimilated when it is in the nascent state...*". We therefore present in this chapter a brief historical survey of the analysis of inhomogeneous solids, with an emphasis on the dominant methods for elastic materials exhibiting continuously variable properties, as is typical of FGM. Note that this survey is illustrative rather than exhaustive. We direct the interested reader to a number of recent reviews on problems associated with inhomogeneous solids and FGM in particular [63, 121, 268, 269, 279].

According to Maugin [169], "*the most common definition of inhomogeneity*[1] *relates to a whole composed of dissimilar or nonidentical elements or parts*". Such microstructure-dependent elements could be situated within that solid for many reasons [168]. For example, defects in polycrystal material structures, changes in the chemical composition of materials, microcracks, inclusions or dislocations, and sudden or continuous physical and chemical actions can all contribute to inhomogeneity. These elements can induce local disturbances of the mechanical fields, which, when consolidated, affect the macroscopic mechanical performance of the solid [32]. Such consolidation (i.e., material *inhomogeneity*) is caused by various effects, which can be divided into three basic types. The first is environmental effects influenced by specific mechanical, physical, or chemical fields or their superposition, radiation, or diffusion of chemically-active agents, gravity, nonuniform temperature, or humidity, etc. The second is technological effects associated with specific treatments during fabrication and exploitation, such as hot-rolling, forging, pressing, quenching, consolidation, and chemical treatment. The third is design intent, such as the situation where inhomogeneity is an intentional result of engineering aimed at reinforcing a material, such as composite materials, FGM, etc.

It seems that Jasinsky (see a survey of his outstanding contribution into the development of theories of elasticity and strength of materials in [281, pp. 294–297]) was the first to evaluate the effect of macro-inhomogeneity resulting from the consolidation of micro-defects in a solid. According to

[1] Note that Maugin has given preference to the rather mathematical term "*inhomogeneous*" over the terms "*nonhomogeneous*" and "*heterogeneous*", which are more widely used in engineering, material and physical sciences.

his hypothesis [119], the material continua can be considered a homogeneous medium when it meets the following empirical criterion:

$$\frac{\mathcal{L}}{\ell} \geq \mathcal{A}^2 , \tag{1.1.1}$$

where $\mathcal{L}$ is the characteristic length of a solid, ℓ denotes the characteristic length of an element representing certain physical properties, and $\mathcal{A}$ is a large real number $(1/\mathcal{A} \ll 1)$ obtained from stochastic experiments. If this criterion is not met, then the material continua cannot be treated as homogeneous, such that we must account for variations in the material properties within the spatial coordinates. Since the initial publication of this theory, a number of criteria have been formulated for more advanced techniques aimed at estimating material inhomogeneity [203]. However, the Jasinsky formula remains both the simplest and most practical.

It is worth noting that there have been many attempts to systemize the types of inhomogeneity in solids based on their physical nature, mechanical behavior, and geometrical shape. Even the term itself has been given different meanings in different studies. The term *inhomogeneity* was used in [149, 217] to define bodies consisting of a number of homogeneous layers (note that Muskhelishvili [186] referred to such solids as compound or piecewise-homogeneous). Lekhnitskii [154] used the term *nonhomogeneous* to refer to both multilayer solids and bodies with continuous variation in elastic properties. In his book [156], he identifies the latter case as *continuously nonhomogeneous*. In other studies, bodies with macro-defects and macro-inclusions are also regarded as inhomogeneous [169]. Such solids are referred to as *statistically* (*or stochastically*) *inhomogeneous* in cases where the defects are randomly distributed [142, 189] within a solid or its parts.

In cases where the inhomogeneity is associated with certain kinds of impact, the consolidation of defects may induce unidirectional distributions. This is termed *inductive inhomogeneity* [142], and includes three basic types: *i*) multilayer or stratified bodies (material properties are constant in certain layers or blocks of a solid); *ii*) continuously inhomogeneous bodies (the variation profiles of the material properties are continuous but not necessarily smooth); and *iii*) multi-modular bodies (the properties are different under contraction and tension). Often, inductively inhomogeneous solids exhibit anisotropic behavior. More detail on the classifications of inhomogeneities can be found in [103, 142, 202].

An important class of inhomogeneity is concerned with the response of certain materials to non-uniform temperature distributions, which

significantly affect mechanical performance. This effect is usually referred to as *thermosensitivity*, and presents a profound impediment to analysis, due to the fact that heat-conduction and thermoelasticity in thermosensitive materials appear to be nonlinear [107, 195]. Nonlinearity arises not only in the heat-conduction equation (the coefficients of which are dependent on the unknown temperature), but also in the specific types of boundary condition, such as complex heat exchange [151].

One specific type of inhomogeneity is concerned with the effects of time. The material properties of concrete, for example, depend significantly on curing time so that they are non-uniform within the solid (e.g., the surface hardens more quickly). Analogous situations can be observed in the imbibition/drying of porous materials, saturation and material diffusion, and fatigue damage. All of these processes are of the evolutionary type and affect the material properties in a non uniform manner within the solid, such that material inhomogeneity is actually a function of time and/or spatial coordinates.

Inhomogeneity can be also categorized with respect to *i*) the number of variable moduli (whether all or only specific material moduli can be regarded as functions of their spatial coordinates) and *ii*) the direction (whether the inhomogeneity is exhibited in one or more spatial directions). In the case of anisotropic inhomogeneous solids, this type of classification is made more complicated by the large number of independent material moduli and the possibility of changes in anisotropy from point to point.

1.2. History of development

1.2.1. Applications in geomechanics

It appears that most of the pioneering works related to the mechanics of inhomogeneous solids have been related to important problems in the field of geomechanics. Most of those studies were performed at the end of the nineteenth century, and have continued in two main directions: wave propagation in soil deposits (Fig. 1.1), with applications in seismology, and the distribution of forces in massive soil deposits when acted upon by either transient or steady-state localized pressures (Fig. 1.2), i.e., indentation problems with applications in civil engineering.

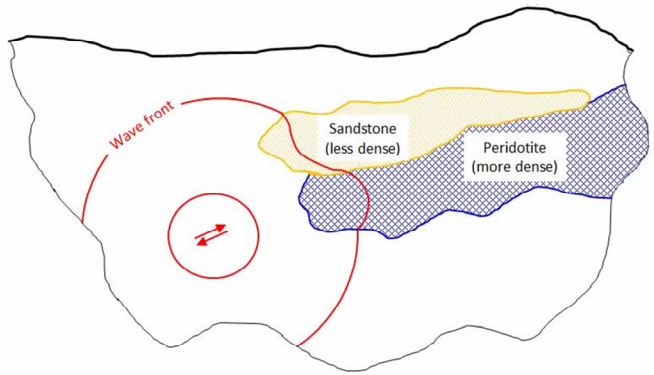

Fig. 1.1 Schematic representation of the soil nonhomogeneity effect in the seismic wave velocity

Due to the stratified nature of soil deposits, most pioneering papers devoted to the analysis of wave propagation have addressed multilayer semi-infinite structures [67, 120, 243]. There are, however, a number of earlier papers that assumed variation in all or certain properties of rocks and soils in a continuous manner with respect to depth coordinate. Specifically, Meissner [172] studied the propagation of surface waves in an inhomogeneous medium, rigidity modulus $G(z)$ and density $\rho(z)$ of which are respectively quadratic and linear functions of depth coordinate z :

$$G(z) = G_0(1+\delta z)^2, \quad \rho(z) = \rho_0(1+\delta z), \quad G_0, \delta = \text{const}. \qquad (1.2.1)$$

Meissner demonstrated that if soil inhomogeneity is taken into account, then the surface waves are purely torsional in character. These waves oscillate horizontally and normally to the direction of propagation, and thus should be regarded as transverse waves. Meissner also demonstrated that these waves exhibit dispersive characteristics, which can be evaluated numerically under the assumption of (1.2.1) using the shear modulus and density given by linear functions of depth. In an attempt to analyze the main features of waves travelling through an inhomogeneous material, Aichi [3] addressed a problem similar to the one considered by Meissner under the assumption that both of the above-mentioned parameters can be represented using exponential functions of depth with dissimilar exponent numbers. Sezawa [248] extended the Meissner–Aichi solution to cases involving two dimensions. The propagation of Rayleigh waves in a continuously-inhomogeneous medium was first considered by Stoneley [263], and later by Pekeris [211]. Further developments were presented in

[40, 47, 51–53, 58, 65, 94, 108–110, 122, 161, 180, 187, 193, 245, 254, 256, 264, 270, 338, 340, 341, 349]. The history of this development is detailed in [67, 188].

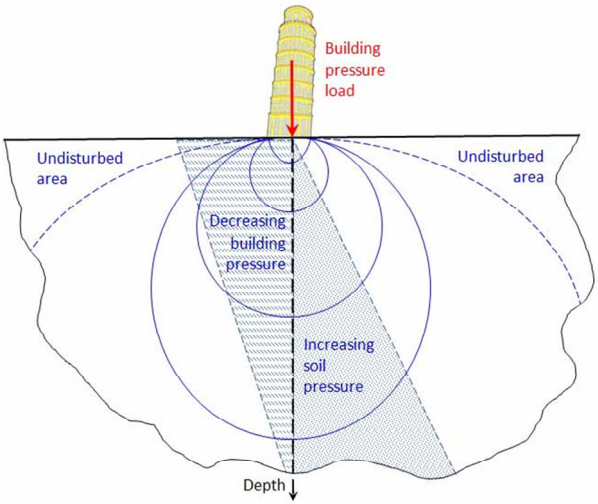

Fig. 1.2 Scheme of pressure distribution in soil due to local pressure

In three papers [42–44], Burmister considered an analogy to the well-known Boussinesq problem to analyze pressure distributions in two- and three-layer soil deposits. Later, Kogan [140] extended this technique to the case of an arbitrary number of layers resting on a homogeneous substrate. A natural development of this technique has been done by Lekhnitskii [155] generalizing the solution for multilayer solids in order to analyze continuous inhomogeneity.

The effect of continuous material inhomogeneity in distribution of pressure in soil deposits due to external force loadings applied to a part of plane boundary or when the boundary suffers a local deflection, has been earlier studied by Fröhlich [74], who has pointed out the importance of encountering the with-the-depth variation of the Young modulus in the evaluation of stress in semi-infinite elastic foundations with a reference to earlier experimental results published by Föppl [72] in 1897. Fröhlich further evaluated the influence of void ratio on variations in the modulus of elasticity in soil deposits modeled by a half-space and attempted thereby the inhomogeneity contribution into the distribution of normal and shearing forces under the plane boundary of a half-space. By making use

of these results, Ohde [201] estimated the depth-variation of the shearing modulus $G(z)$ of an elastic soil foundation as

$$G(z) = g(z + z_0)^w, \tag{1.2.2}$$

with g, w, and z_0 being constants.

Besides the dependence (1.2.2), there were a number of similar estimations for the dependences of elastic moduli on the depth coordinate. For example, Lekhnitskii [155] demonstrated that if the Poisson ratio is assumed to be constant and the Young modulus varies proportionally to the depth, then the radial stress σ_{rr} due to a concentrated force applied to a point on the surface can be expressed as follows:

$$\sigma_{rr} = -\frac{2}{\pi r}(P_x + 2P_y\theta)\cos\theta, \tag{1.2.3}$$

where P_x and P_y are the components of applied force in directions perpendicular and parallel to the surface, respectively, and r and θ are the polar coordinates with the origin at the point where the force is applied. It follows from (1.2.3) that if the applied force is normal to the surface (i.e., $P_y = 0$), then the radial stress distribution within the framework of the plane problem for an incompressible isotropic material is the same as for a homogeneous half-plane. If the Young modulus varies inversely proportional to the depth, formula (1.2.3) takes the form:

$$\sigma_{rr} = -\frac{P_x}{\pi x}, \tag{1.2.4}$$

wher x is the depth coordinate. Furthermore, concentrated force applied parallel to the boundary does not cause any stress in the half-plane.

In [80], Gibson addressed the case of linear depth-variation in the sharing modulus $G(z) = G(0) + mz$ in an incompressible elastic foundation $z \geq 0$. He also pointed out other simple cases of inhomogeneity; i.e., $G(z) = G(0)\exp(\lambda z)$ and

$$G(z) = \frac{\eta G(0)}{\eta + z}, \tag{1.2.5}$$

which allow for the comparatively simple analysis of stresses or displacements in such solids. Here, m, λ, and η are constant parameters and $G(0)$ denotes the value of the shearing modulus on surface $z=0$. In [83], these results were extended for the case of an inhomogeneous layer. Awojobi [19] was able to solve the dual integral equations of a mixed boundary value problem by studying vertical vibrations in so-called "Gibson soil" (i.e., the kind of inhomogeneity addressed by Gibson). In [20], the same technique was used for the analysis of multilayer coatings resting on a homogeneous substrate. "Gibson soil of second kind" (i.e., soil with the shear modulus given in (1.2.5) with negative η) was analyzed in [21]. Those results were further developed in [22, 38, 39, 82, 84, 86, 253]. Gibson and Sills [85] also analyzed the case where the Poisson ratio varies with depth.

In addition to the distribution of stresses within the depth of soil deposits, another important problem in geomechanics is determining contact pressure on the surface of a half-space (or a half-plane) caused by an indenter applied against the surface or surface deflection resulting from the imposed pressure. In a series of papers, Popov (e.g., [221]) introduced the following simple formula for the calculation of surface deflection $w(x,y)$ due to a force $p(x,y)$ applied to boundary $z=0$ of half-space $z \geq 0$:

$$w(x,y)=\int_0^\infty f_0(t)dt \iint_{-\infty}^{\infty} J_0\left(\sqrt{(x-\xi)^2+(y-\eta)^2}\right)p(\xi,\eta)d\xi d\eta, \quad (1.2.6)$$

where $J_0(x)$ is the zero order Bessel function of the first kind. In view of (1.2.6), the problem of determining surface deflection is reduced to determining function $f_0(t)$. Popov demonstrated that if the Young modulus of the half-space varies as a power function of depth (i.e., $E(z)=E_n z^n$, $E_n=\text{const}$, $n=\text{const}$ and $0<n<1$), then function $f_0(t)$ can be described in the following form:

$$f_0(t)=\frac{\alpha}{\pi E_n}\frac{\Gamma\left(\frac{1}{2}-\frac{\nu}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}\right)}\left(\frac{t}{2}\right)^n . \quad (1.2.7)$$

Here, constant $\alpha = \dfrac{3+n}{2(1+n)(2+n)}$ was contributed by Klein [137], who was the first to derive the following equation for surface deflection in this type of inhomogeneity:

$$w(x,y) = \frac{\alpha}{\pi E_n} \iint_{-\infty}^{\infty} \frac{p(\xi,\eta)d\xi d\eta}{\left((x-\xi)^2 + (y-\eta)^2\right)^{\frac{1+m}{2}}} \quad . \tag{1.2.8}$$

This equation can also be obtained by inserting (1.2.7) into (1.2.6). A similar result was found for the case of exponential variation in the Young modulus (i.e., $E(z) = E_\gamma \exp(\gamma z)$, $E_\gamma = \text{const}$ and $\gamma = \text{const}$ [220]).

The analysis of surface deflections and inner stresses for certain dependences of the Young modulus on depth and various loading profiles $p(x,y)$ have been developed using equations similar to (1.2.8) by Rostovtsev [234, 235], Rostovtsev and Khranevskaia [237], Mossakovskii [185], Chuaprasert and Kassir [56, 131], Kassir [128–130], Carrier and Christian [46], Chuong [57], and many others.

1.2.2. Mechanics of composite materials

The analysis of inhomogeneous materials was developed further in the mid-twentieth century due to advances in materials science and the widespread implementation of composite materials. Composite materials may exhibit anisotropic [55, 154, 156, 277, 310] and/or inhomogeneous [103, 104, 202, 203] mechanical behavior [62, 87, 123, 194], contingent on the methods used to combine constituent phases or reinforcing elements. In the analysis of material inhomogeneity, one important publication of note is a collection of papers [203] presented at *The Warsaw Symposium on Nonhomogeneity in Elasticity and Plasticity (Warsaw, Poland, September 3 – 9, 1958)*. This book reflects the experience of numerous scientists from 14 countries and summarizes the basic methods developed for the analysis of inhomogeneous materials in the fields of elasticity, plasticity, rheology, dynamics, and wave propagation as well as the statistical methods used to characterize micro-non-homogeneity.

It is worth noting that the periodic or semi-periodic material structures found in some composite materials (e.g., fiber composites and reinforced composites) make it possible to nullify the effects of material inhomogeneity through the implementation of various homogenization procedures, such that the focus is shifted solely to the effects of anisotropy

[25]. In many cases however, inhomogeneity must be considered within the context of multilayer structures [217, 258, 259].

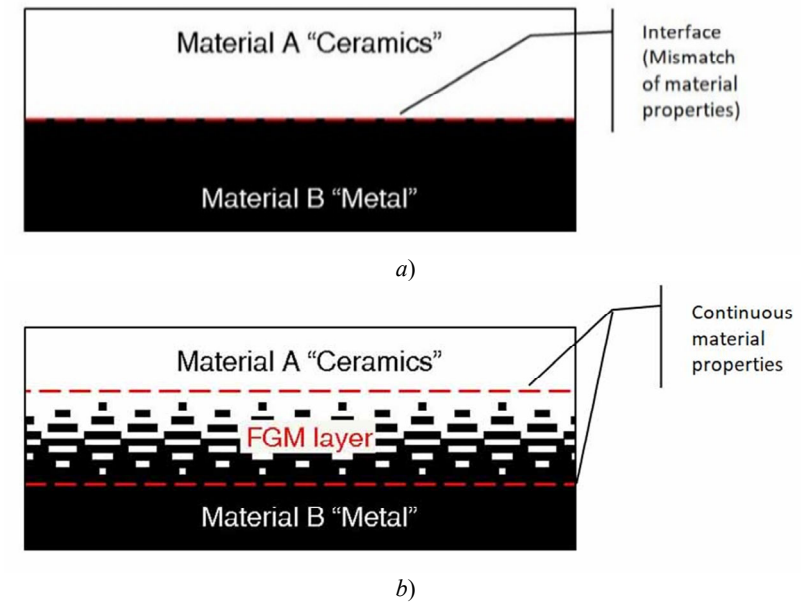

Fig. 1.3. Two-component metal-ceramic composite without (*a*) and with (b) FGM interlayer

1.2.3. Functionally-graded materials

In the 1980s, considerable advances were made in the fabrication of inhomogeneous materials with intentionally-continuous variations in macroscopic material properties based on desired distribution profiles [141, 190, 165]. Much of this work focused on functionally-graded materials (FGM) – a class of multiphase composite combining two or more phase-materials with contrasting properties. According to Rabin and Shiota [231], "*the term FGM, now widely used by the materials community, originated in Japan in the late 1980s as a description of a class of engineering materials exhibiting spatially inhomogeneous microstructure and properties*" (see also [115, 141, 183, 230, 233]). FGM are widely used to improve the operational performance of structural members subjected to mechanical as well as thermal loading [182, 195]. They comprise dissimilar materials that provide high thermal and mechanical resistance (e.g., ceramics and metals) [33, 54, 132, 184, 255,

266, 339]. Residual stresses that develop at the interfaces between the two constituents due to the mismatch in material properties can lead to material degradation (Fig 1.3*a*). This effect can be minimized by continuously (or almost continuously) varying the material properties of FGMs (Fig 1.3*b*) from one constituent material to another [16, 196, 271]. This type of variation is generally presented in the form of arbitrary dependences of the elastic moduli on spatial coordinates. Note that this makes it nearly impossible to solve problems of elasticity and thermoelasticity analytically [272], due to the fact that the governing equations include unknown variable coefficients. Thus, the development of efficient methods applicable to the thermomechanical analysis of inhomogeneous materials (and FGM in particular) is an important problem in modern engineering. The interested reader can find recent reviews of problems related to FGM solids in [63, 121, 268, 269, 279].

1.3. Overview of Solutions

1.3.1. Specific solution methods

Modeling and analysis methods vary according to the type of inhomogeneity. The methods also differ in terms of body shape, coordinate system, the type of solution that is required, techniques used, and loading types. There is really no way therefore to present a comprehensive review of methods used in the analysis of inhomogeneous solids. Thus, in this section, we present a brief description of the methods and solutions that are most relevant to the subject matter dealt with in this book. We are aware that many important results remain beyond the scope of this review, and for this we apologize.

Mikhlin [177] developed a general theory of the hyperbolic equations with variable (piecewise-variable) coefficients used to analyze the dynamic processes in non-homogeneous media with further development, e.g., in [23]. The method of plane waves was presented in [61, 153] aimed at dealing with systems that involve hyperbolic equations with smooth variable coefficients. An extensive review of early work in this area can be found in [24]. This general theory makes it possible to derive important theoretical results, such as the mathematical substantiation of the existence and uniqueness of solutions for certain functional spaces; however, it has not found widespread use in practical engineering applications.

The complex variable method was developed by Mishiku and Teodosiu [181] for the analysis of plane static problems, which were reduced to a set

of conjugation problems to be solved through successive approximations. Gorbachev and Pobedria [92, 93] used the averaging method for differential equations by reducing problems of elasticity in inhomogeneous materials to recurrent problems for homogeneous bodies that approximate but satisfy the boundary conditions. Naumov and Chistyak [191] constructed a formal asymptotic solution to the problem of an arbitrarily inhomogeneous layer. Shevchuk [251] presented a method by which to derive approximate solutions of heat conduction problems in solids with thin multilayer coatings modeled using generalized boundary conditions.

A number of solutions have been constructed using approximate formulations [232] and variational principles [1, 30, 340]. The perturbation method was proposed in [150] to reduce thermoelasticity problems for thermosensitive bodies to a recurrent sequence of boundary-value problems to be solved using differential equations with constant coefficients. The Hamilton variational principle was presented in [102] to develop a hybrid numerical method for the analysis of transient wave propagation in an FGM cylinder. The simplified Gurtin's variational principle was presented in [347] for FGM thermoviscoelastic plates. Variational principles were also used to develop homogenization procedures for micro-inhomogeneous periodic structures [15, 163]. Vasilenko [309] proposed a numerical approach to solving a non-axisymmetric problem for a radially-inhomogeneous anisotropic cylinder. Klimenko [138] reported a numerical solution for a cylinder inhomogeneous in the circumferential direction. Other important methods include the finite element method [91, 171, 276, 307], the boundary element method [95, 97, 113, 350], the method of the displacement potential [127, 212], and the finite difference method [66].

One interesting analytical-numerical technique was developed by Aizikovich et al. [4, 5, 7] for the analysis of contact problems in inhomogeneous materials with arbitrary variations in the properties with respect to depth. This technique is based on the bilateral asymptotic method [8]. The key point is to numerically evaluate the kernels of the obtained integral equations. After the kernel structure is defined, it can be approximated using a special expression, which makes it possible to solve the integral equation analytically. This allows for the calculation of an analytical solution, which is convenient for mechanical analysis of the effects of arbitrary inhomogeneity [147, 148] and FGM material properties using indentation experiments [6, 9]. In the following, we discuss the dominant analytical methods used in the analysis of continuously-inhomogeneous solids.

1.3.2 Dominant methods

1.3.2.1. Material moduli in form of elementary functions of spatial coordinates

The dominant approach to the analysis of continuously-inhomogeneous solids is based on the assumption that the material properties take the form of specific elementary functions (e.g., linear, polynomial, or exponential functions) in a manner that allows for the separation of variables in the governing equations. This yields comparatively simple solutions based on classical techniques. Various issues involved in the separation of variables by means of this approach have been discussed by Kolchin [142], Leknitskii [156], Teodorescu [278], and others.

In illustrating this technique, we consider a plane-stress problem for an elastic element $-a \le x \le a$, $-b \le y \le b$, where x and y are dimensionless Cartesian coordinates and a and b are constant parameters. By introducing the potential Airy function $\varphi(x,y)$ [282], this problem can be reduced to the following equation [288]:

$$\nabla^2\left(\frac{\nabla^2\varphi(x,y)}{E(x,y)}\right)-2\frac{\partial^2\varphi(x,y)}{\partial x\partial y}\frac{\partial^2}{\partial x\partial y}\left(\frac{1+\nu(x,y)}{E(x,y)}\right)$$

$$-\frac{\partial^2\varphi(x,y)}{\partial y^2}\frac{\partial^2}{\partial x^2}\left(\frac{1+\nu(x,y)}{E(x,y)}\right)$$

$$-\frac{\partial^2\varphi(x,y)}{\partial x^2}\frac{\partial^2}{\partial y^2}\left(\frac{1+\nu(x,y)}{E(x,y)}\right)=0\,, \qquad (1.3.1)$$

where ∇^2 stands for the differential Laplace operator. If we assume that the Young modulus is $E=E_0\varsigma(y)$ (where E_0 is a dimensional constant and $\varsigma(y)$ is an arbitrary twice-continuously-differentiable function) and Poisson ratio ν is a constant, then the problem on the elastic equilibrium of the considered element can be reduced to the following equation:

$$\nabla^4\varphi(x,y)-2\frac{1}{\varsigma(y)}\frac{d\varsigma(y)}{dy}\frac{\partial}{\partial y}\left(\nabla^2\varphi(x,y)\right)$$

$$+\left(\frac{1}{\varsigma(y)}\frac{d^2\varsigma(y)}{dy^2}-2\left(\frac{1}{\varsigma(y)}\frac{d\varsigma(y)}{dy}\right)^2\right)$$

$$\times\left(\nu\frac{\partial^2\varphi(x,y)}{\partial x^2}-\frac{\partial^2\varphi(x,y)}{\partial y^2}\right)=0\,. \tag{1.3.2}$$

Note that in the plane-strain case, we substitute ν with $\nu/(1-\nu)$ and ς with $\varsigma/(1-\nu^2)$.

Clearly, this equation has constant coefficients under the following condition:

$$\frac{1}{\varsigma(y)}\frac{d\varsigma(y)}{dy}=\kappa=\text{const}\,. \tag{1.3.3}$$

This means that

$$\varsigma(y)=\varsigma_0\exp(\kappa y)\,, \tag{1.3.4}$$

where $\varsigma_0>0$ is an arbitrary constant.

If we assume that

$$\frac{1}{\varsigma(y)}\frac{d^2\varsigma(y)}{dy^2}-2\left(\frac{1}{\varsigma(y)}\frac{d\varsigma(y)}{dy}\right)^2=0\,, \tag{1.3.5}$$

then we obtain the following:

$$\varsigma(y)=\frac{1}{\varsigma_1+\varsigma_2 y}\,, \tag{1.3.6}$$

where ς_1 and ς_2 are arbitrary constants that ensure the feasibility of the Young modulus; i.e., $\varsigma_1+\varsigma_2 y>0$ for $-b\le y\le b$. This expression does not make the coefficients in (1.3.2) constant; however, it does allow for the representation of (1.3.2) in the following form:

$$(\varsigma_1+\varsigma_2 y)\nabla^4\varphi(x,y)+\varsigma_2\frac{\partial}{\partial y}\left(\nabla^2\varphi(x,y)\right)=0\,, \tag{1.3.7}$$

which can be integrated (e.g., for the case where φ is a harmonic function), as follows:

$$\nabla^2 \varphi(x, y) = 0 . \tag{1.3.8}$$

Note that equation (1.3.7) does not involve the Poisson ratio ν, that is typical for the plane problems of elasticity in homogeneous materials with no body forces or thermal loading [282].

Another type of material-distribution profile that allows for the integration of equation (1.3.2) by making use of the classical methods has the form of a power function

$$\varsigma = \varsigma_3 y^{\kappa_0} , \tag{1.3.9}$$

where ς_3 and κ_0 are arbitrary constants.

For obvious reasons, i.e., a comparatively simple solution technique along with the ability to implement classical methods, representations (1.3.4), (1.3.6), and (1.3.8), with some modifications, have remained in the spotlight from the very beginning till nowadays. As mentioned in Section 1.2.1, Gibson [80, 81] presented a simple approach to the analysis of a nonhomogeneous half-space with the shear modulus in the forms given in (1.3.4) and (1.3.6), as well as the linear form for problems in geomechanics. In [21, 46, 253], these types of nonhomogeneity were examined using numerical techniques with different loadings on the half-space boundary. Korenev [143], Mossakovsii [185], Popov [222], and many others have presented solutions to problems involving indentation of a circular punch into an exponentially-nonhomogeneous half-space using the couple-integral equation representation. Mixed and contact problems for a nonhomogeneous half-space with power-law dependences of the elastic moduli on the depth-coordinate were addressed in [155, 234, 236]. Giannakopoulos and Pallot [79] presented an exact solution to the axisymmetric problem of indentation by a circular punch into an elastic half-space, where the Young modulus varies with depth in accordance to the power law and Poisson's ratio is constant. The same material properties were considered in [28] for the analysis of a plane-strain contact problem with an inhomogeneous half-space subjected to the action of a rigid punch within the finite area of its limiting plane. Teodorescu [278, p. 653] generalized the representation (1.3.4) by considering inhomogeneity in the following form:

$$\varsigma = \varsigma_0 \exp\left(f(x, y, z) \right) , \tag{1.3.10}$$

where $f(x, y, z)$ is a continuous differentiable function of class C^4. An extensive review of studies involving exponential inhomogeneity can be found in [302].

Similar types of inhomogeneity have been analyzed in a wide range of coordinate systems. Zimmerman and Lutz [351] used the Frobenius series method to analyze thermal stresses in an FGM cylinder under uniform heating to characterize the linear dependences of properties on the thickness-coordinate. Horgan and Chan [111, 112] constructed a solution to an isotropic inhomogeneous hollow circular cylinder and disk, where the Young modulus is a power function of the thickness coordinate under constant rotation velocity and uniformly pressurized on inner and outer boundaries. They reduced this problem to the Navier equation for radial displacement to be solved in a closed form. A closed-form solution was also reported by Oral and Anlas [205] for a hollow orthotropic cylinder with analogous variation in the elastic moduli. Jabbari et al. [117] solved plane axi-symmetric elasticity and thermoelasticity problems for hollow cylinders by reducing them to the governing Navier equation. An analytic solution to the latter equation for plane non-axisymmetric elasticity and thermoelasticity problems in a thick hollow cylinder was presented [118] in the form of the complex Fourier series where the material properties are given as power functions of the radial coordinate. The non-axisymmetric temperature has also been derived for cases with a various thermal conduction coefficients. A similar solution was reported by Tarn [274] and Tarn and Chang [275] for a radially-inhomogeneous piezoelectric circular cylinder. Zhang and Hasebe [346] constructed a solution to the plane non-axisymmetric elasticity problem for a radially-inhomogeneous hollow cylinder with an exponential Young's modulus and constant Poisson's ratio.

At this point, the sheer number of studies on this method makes it difficult to present an exhaustive review (numerous references are listed in [63, 121, 269, 279, 335]). The popularity of this approach can be attributed to several advantageous features. First, the assumption that material properties are specific functions of spatial coordinates makes it possible (in many cases) to simplify the governing equations, such that classical methods of mathematical physics are applicable. Second, this approach allows for the modeling and analysis of inhomogeneity implementing the dependence of elastic moduli on more than one spatial coordinate. This makes it possible to obtain closed-form analytical solutions for use as benchmarks in the verification and validation of solutions applicable to problems of greater complexity.

Nonetheless, the solutions obtained using this approach are subject to limitations. First, they are not widely generalizable; therefore, a new