Theory and Methods of Vector Optimization (Volume One)

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By

Yu. K. Mashunin

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MAIN DESIGNATIONS

N – set of natural numbers.
R - set of real numbers; numerical straight line.
R^n – arithmetic real n-dimensional space;
Euclidean n-dimensional space;
$\{a, b, c, x, y, \ldots\}$ – the set consisting of elements $a, b, c, x, y, \ldots$
$\forall$ - generality quantifier: "for all."
$\exists$ - existential quantifier: "exists."
$\varnothing$ - empty set.
$\in$ - sign of an accessory to a set.
$\subset$ - sign of an inclusion of a set.
$\boldsymbol{A}\cap\boldsymbol{B}$ – a product of the sets of ***A*** and ***B***.
$\boldsymbol{A}\cup\boldsymbol{B}$– a union of the sets of ***A*** and ***B***.
$\boldsymbol{X}=\{x_1,\ldots, x_j,\ldots, x_N\}$ – the set consisting of ***N*** elements or
$\boldsymbol{X}=\{x_j, j=1, \ldots, N\}$, or $\boldsymbol{X}=\{x_j, j=\overline{1,N}\}$, where j – number of the index (object), N – number (number of the last index), ***N*** – a set of indices [1].
$\equiv$ - identically equal.
lim – a limit.
max X – the greatest (maximal) element of a great number of X.
min X – the least (minimum) element of a great number of X.
$\max\limits_{x\in X} f(X)$ - the greatest (maximal) value of function f on a great number of X.
$\min\limits_{x\in X} f(X)$ - the least (minimum) value of function f on a great number of X.
$\max\limits_{x\in X} f(X)\equiv F$ - the greatest (maximal) value of function f to which the functional value F, on a great number of X is identically appropriated.

[1] Each number, for example 100=N, can designate a set of ***N***, and an element in this set with number N is allocated with writing (the set is designated by the aliphatic and sloping code - ***N***).

$\min\limits_{x \in X} f(X) \equiv F$ - the least (minimum) value of function *f* to which the functional value *F*, on a great number of *X* is identically appropriated.

max $\boldsymbol{F}(\boldsymbol{X})$={max $\boldsymbol{f_k}(\boldsymbol{X})$, $\boldsymbol{k}=\overline{1,K}$ } - vector criterion of maximizing with which each component is maximized, *K* – number, a $\boldsymbol{K}\equiv\overline{1,K}$ - a set of criterion indices.

max $\boldsymbol{F}_1$(X)={max $\boldsymbol{f_k}(\boldsymbol{X})$, $\boldsymbol{k}=\overline{1,\mathrm{K}_1}$ } - vector criterion of maximizing with which each component is maximized, $\boldsymbol{K}_1$ – number, a $\boldsymbol{K}_1\equiv\overline{1,\mathrm{K}_1}$ - a subset of indices of criterion of $\boldsymbol{K}_1\subset\boldsymbol{K}$.

min $\boldsymbol{F}_2(\boldsymbol{X})$={$\boldsymbol{f_k}(\boldsymbol{X})$, $\boldsymbol{k}=\overline{K_1+1,K}$ } - vector criterion of minimization, $\boldsymbol{K}_2\equiv \overline{K_1+1,K}\equiv\overline{1,K_2}$ - a subset of indices of criterion, $\boldsymbol{K}_2$ – number, $\boldsymbol{K}_2\subset\boldsymbol{K}$, $\boldsymbol{K}_1\cup\boldsymbol{K}_2=\boldsymbol{K}$.

INTRODUCTION

At the beginning of the 20th century, during research into commodity exchange, Vilfredo Pareto [1] mathematically formulated the criterion of optimality, the purpose of which is to estimate whether the proposed change improves common welfare in an economy. Pareto's criterion claims that any change which does not inflict loss on anyone and brings benefit to some is an improvement.

Despite some imperfections, Pareto's criterion broadly makes sense (e.g., in the creation of development plans for an economic system when the interests of its constituent sub-systems or groups of economic objects are considered). According to Pareto's theory, the distribution of resources is optimum for the conditions of a perfect competitive market structure. In other words, perfect competitive markets guarantee that an economy will automatically reach points of optimality. However, the distribution of resources being Pareto-optimal does not always mean they are socially optimal, as a society can choose (by means of the state's economic policy) to limit any point's accessible usefulness, and it may or may not be the point answering to social optimality. Resources can be effective (according to Pareto) if distributed, even in situations of extreme inequality. This is promoted, as a rule, by the economic policy directions of the state which provides benefits to one group of the population at the expense of another.

The Pareto criterion was later transferred to optimization problems with a set of criteria, where problems were considered in which optimization meant improving one or more indicators (criteria), provided that others did not deteriorate. Multi-criteria optimization problems arose. As a rule, a set of criteria was represented as a vector of criteria (hence vector optimization problems or vector problems in mathematical programming (VPMP)). It immediately became clear that there were optimum points of Pareto in VPMP but, as a rule, there were fewer of these than sets of admissible points.

Further interest in the problems of vector optimization increased in connection with the development and widespread use of computer technology in the work of economists and mathematicians. The functioning of the majority of economic systems depends on a set of indicators (criteria), i.e., the substance of economic systems includes multi-criteria and only the lack of mathematical methods in solving the problems of vector optimization (the cornerstone of the specified models) has constrained their use, both in theory and in practice.

It later became clear that multi-criteria (vector) problems arose not only in the economy, but also in technology, e.g., in the projection of technical systems, the optimum projection of chips, and in military science.

The solution to the problem of vector optimization creates a number of difficulties and, apart from their conceptual character, the principal aim of them is to understand what it means to solve a problem of vector optimization (i.e., to create the principle of optimality showing why one decision is better than another, and defining a choice rule for the best decision). This aim of this book is to find a solution to this problem.

This monograph will present a systemic analysis of the theory and methods of vector optimization and the practical applications, first in the modelling and forecasting of the development of economic systems, and secondly in problems of projection and the modelling of technical systems. These models are used during the development and adoption of management decisions, on the basis of what is developed in the MATLAB system (the software used in solving linear and non-linear vector problems).

The monograph includes two volumes. The first volume, *The Theory and Methods of Vector Optimization*, includes seven chapters and considers the methods for solving vector problems in linear and non-linear programming. The main focus is on the author's theory and methods of vector optimization. Bases and design methods for solving vector (multi-criteria) problems in mathematical programming are also presented. The main difference from the standard approaches to solving vector problems is that they are constructed on axiomatics and the principles of an optimal solution. This demonstrates the way in which one decision is more optimum than another. As the decision is carried out on a set of several (system) criteria, the system analysis and systemic decision-making, in total, are put into the decision algorithm. The algorithm allows us to solve linear and non-linear problems with equivalent criteria and at the given priority of the criterion. Theoretical issues with the duality of vector problems in linear programming are investigated, and the interrelation between the theory of the adoption of an administrative decision and vector optimization is also presented. Problem-solving in decision-making is shown under conditions of certainty and uncertainty. The majority of mathematical methods are followed not only by concrete numerical examples but also by their categorisation in the MATLAB system.

The second volume, *Vector Optimization Modelling of Economic and Technical Systems*, presents the practical use of the theory and methods of vector optimization in the field of mathematical modelling and simulations of economic and technical systems. The second volume of the work is divided into two parts: economic systems; and technical systems.

In the first part of the second volume, "Vector Optimization in the Modelling of Economic Systems," the following is considered: the theory, modelling, forecasting and adoption of administrative decisions at the level of production, market and regional systems. This is divided into three chapters where questions around the creation of mathematical models at the level of the firm, the market and the region are explored. An analysis is carried out of the "theories of the firm", on the basis of which the mathematical model of prediction and decision-making across many criteria (purposes) of the development of the firm is constructed. Modelling and the adoption of production decisions on the basis of such models can be carried out for small, medium-sized and major companies. At the level of the market, a mathematical model is constructed which includes the purposes of all consumers and producers in total, in the form of a vector problem in linear programming. The constructed mathematical market model allows research to be conducted into the structure of the market and helps make decisions while taking purposefulness into account. Such a model resolves issues of equality of supply and demand in the dynamics of a competitive economy. At the level of the region, a mathematical model is constructed which includes economic targets for all sectors of the region and defines the dynamics of the development of the regional economy within investment processes.

In the second part, "Vector Optimization in the Modelling of Technical Systems," questions are considered around the theory, modelling, development practice and adoption of management decisions in technical systems. This is presented in three chapters. The complexity of modelling technical systems is defined by the fact that, in the functioning of a technical object, a system is defined by a set of characteristics that depend on the parameters of a technical system. An improvement in one of these characteristics leads to a deterioration in another. There is a problem in determining such parameters which would improve all functional characteristics of the technical system at the same time, i.e., the solution to a vector (multi-criteria) problem is necessary.

These problems are now being solved at both technical (experimental) and mathematical (model) levels. The costs associated with the experimental level are much higher than those at the information level. The methods being offered also solve this problem. The concept of optimal design in a technical system is developed, as well as the organization of this under conditions of certainty and uncertainty. Theoretical modelling problems are accompanied by numerical simulations of technical systems.

The two volumes of this monograph are based on research and analysis of similar literature in the field of the theory and methods of vector optimization (1-126). The first volume is constructed on the basis of research into foreign literature (1, 2, 26-59), domestic authors (3-10), and the author's own developments (11-23). In the second volume, analysis and research are conducted first in the field of economics, i.e., the theory of the firm, market theory, and decision-making in the regional economy (59-96); secondly in the field of technical research and decision-making (97-126); and thirdly from experience of teaching both the theory of management and the development of management decisions at Far Eastern Federal University.

The book is designed for students, graduate students, scientists and experts dealing with theoretical and practical issues in the use of vector optimization, the development of models and predictions of developments in economic systems, and the designing and modelling of technical systems.

VOLUME 1. THE THEORY AND METHODS OF VECTOR OPTIMIZATION

This volume presents a theory and methods for solving vector optimization problems; the common difficulties surrounding the definition of vector optimization; a development of the axiomatics of vector optimization on the basis of which the principles of optimality in solving vector problems are formulated; a consideration of the theoretical questions related to the principles of optimality; methods for solving vector problems in mathematical programming, allowing for solutions at equivalent criteria and with the given prioritized criterion; and an investigation into the theory of duality in vector problems of linear programming.

Further, the Appendix presents a comparison of the known approaches with the developed method, which is based on a normalization of criteria and the principle of a guaranteed result.

CHAPTER 1

VECTOR PROBLEMS IN MATHEMATICAL PROGRAMMING (VPMP)

1.1. Problems in defining vector optimization

A vector problem in mathematical programming (VPMP) is a standard mathematical-programming problem including a set of criteria which, in total, represent a vector of criteria.

It is important to distinguish between uniform and non-uniform VPMP: a uniform maximizing VPMP is a vector problem in which each criterion is directed towards maximizing; a uniform minimizing VPMP is a vector problem in which each criterion is directed towards minimizing; a non-uniform VPMP is a vector problem in which the set of criteria is shared between two subsets (vectors) of criteria (maximization and minimization respectively), e.g., non-uniform VPMP are associated with two types of uniform problems.

According to these definitions, we will present a vector problem in mathematical programming with non-uniform criteria [11, 27] in the following form:

$$Opt\ F(X) = \{max\ F_1(X) = \{max\ f_k(X),\ k = \overline{1, K_1}\}, \tag{1.1.1}$$

$$min\ F_2(X) = \{min\ f_k(X),\ k = \overline{1, K_2}\}\}, \tag{1.1.2}$$

$$G(X) \leq B, \tag{1.1.3}$$

$$X \geq 0, \tag{1.1.4}$$

where $X=\{x_j,\ j=\overline{1,N}\}$ - a vector of material variables, N-dimensional Euclidean space of R^N, (designation $j=\overline{1,N}$ is equivalent to j=1,...,N);

$F(X)$ - a vector function (vector criterion) having K – a component functions, (K - set power $\boldsymbol{K}$), $F(X) =\{f_k(X),\ k=\overline{1,K}\}$. The set $\boldsymbol{K}$ consists of sets of $\boldsymbol{K}_1$, a component of maximization and $\boldsymbol{K}_2$ of minimization; $\boldsymbol{K}=\boldsymbol{K}_1\cup\boldsymbol{K}_2$ therefore we enter the designation of the operation "*opt,*" which includes *max* and *min* (a definition of the operation *"opt"* is given in section 2.3);

$F_1(X) = \{f_k(X), k=\overline{1, K_1}\}$ – maximizing vector-criterion, K_1 – number of criteria, and $\boldsymbol{K}_1 \equiv \overline{1, K_1}$ - a set of maximizing criteria (a problem (1.1.1), (1.1.3), (1.1.4) represents VPMP with the homogeneous maximizing criteria). Let's further assume that $f_k(X), k=\overline{1, K_1}$ - the continuous concave functions (we will sometimes call them the maximizing criteria);

$F_2(X)=\{f_k(X), k=\overline{K_1+1, K}\}$ - vector criterion in which each component is minimized, $\boldsymbol{K}_2 \equiv \overline{K_1+1, K} \equiv \overline{1, K_2}$ - a set of minimization criteria, $\boldsymbol{K}_2$ – number, (the problems (1.1.2)-(1.1.4) are VPMP with the homogeneous minimization criteria). We assume that $f_k(X), k=\overline{K_1+1, K}$ - the continuous convex functions (we will sometimes call these the minimization criteria), i.e., $\boldsymbol{K}_1 \cup \boldsymbol{K}_2 = \boldsymbol{K}$, $\boldsymbol{K}_1 \subset \boldsymbol{K}$, $\boldsymbol{K}_2 \subset \boldsymbol{K}$.

$G(X) \leq B$, $X \geq 0$ - standard restrictions, $g_i(X) \leq b_i$, $i=1,...,M$ where b_i - a set of real numbers, and $g_i(X)$ are assumed continuous and convex.

$\boldsymbol{S} = \{X \in \boldsymbol{R}^N \mid X \geq 0, G(X) \leq B\} \neq \varnothing$, where the set of admissible points set by restrictions (1.1.3)-(1.1.4) is not empty and represents a compact.

where $X=\{x_j, j=\overline{1, N}\}$ - a vector of material variables, N-dimensional Euclidean space of R^N, (designation $j=\overline{1, N}$ is equivalent to $j=1,...,N$);

$F(X)$ - a vector function (vector criterion), $F(X)=\{f_k(X), k=\overline{1, K}\}$. The set $\boldsymbol{K}$ consists of sets of $\boldsymbol{K}_1$, a component of maximizing and $\boldsymbol{K}_2$ of minimization; $\boldsymbol{K}=\boldsymbol{K}_1 \cup \boldsymbol{K}_2$ therefore we enter the designation of the operation "*opt*" including *max* and *min*; $F_1(X) = \{f_k(X), k=\overline{1, K_1}\}$ - vector maximizing criterion, K_1 – number of criteria, and $\boldsymbol{K}_1 \equiv \overline{1, K_1}$ - a set of indexes of criterion; $\boldsymbol{K}_2 \equiv \overline{K_1+1, K} \equiv \overline{1, K_2}$ - vector minimization criterion. $\boldsymbol{K}_1 \cup \boldsymbol{K}_2 = \boldsymbol{K}$, $\boldsymbol{K}_1 \subset \boldsymbol{K}$, $\boldsymbol{K}_2 \subset \boldsymbol{K}$.

$G(X) \leq B$, $X \geq 0$ - standard restrictions, $g_i(X) \leq b_i$, $i=1,...,M$ where b_i - a set of real numbers, and gi(X) are assumed continuous and convex.

$\boldsymbol{S} = \{X \in \boldsymbol{R}^N \mid X \geq 0, G(X) \leq B\} \neq \varnothing$, where the set of admissible points set by restrictions (1.1.3)-(1.1.4) is not empty and represents a compact.

The vector minimization function (criterion) $F_2(X)$ can be transformed to the vector maximization function (criterion) by the multiplication of each component of $F_2(X)$ to minus unit. The vector criterion of $F_2(X)$ is injected into VPMP (1.1.1)-(1.1.4) to show that, in a problem, there are two subsets of criteria of $\boldsymbol{K}_1$, $\boldsymbol{K}_2$ with, in essence, various directions of optimization.

We assume that the optimum points received by each criterion do not coincide for at least two criteria. If all points of an optimum coincide among themselves for all criteria, then we consider the decision trivially.

1.2. A case study of vector optimization

The information analysis on VPMP (1.1.1)-(1.1.4) that we will carry out assumes there is a possibility of the decision being made separately for each component of vector criterion.

1) We show that exact upper and lower boundaries exist for any of the criteria for all $\forall k \in \boldsymbol{K}$ on the admissible set $\boldsymbol{S}$.

So, really:

a) In accordance with the Weierstrass theorem on the achievement of a continuous function, on the compact of its exact faces for the set of admissible points $\boldsymbol{S} \neq \varnothing$ an optimum (best point) exists for each k-th component ($k = \overline{1, K}$) of the vector criterion, i.e., $X^*_k \in \mathbf{R}^N$ is the optimum point and the value of the objective function (criterion) at this point: $f^*_k \equiv f_k(X^*_k)$ will be obtained to solve VPMP (1.1.1) - (1.1.4) separately for each component of the vector criterion, with $k \in \boldsymbol{K}_1$, which of course are solved for a maximum; and for criteria with $k \in \boldsymbol{K}_2$ - at a minimum.

b) According to the same Weierstrass theorem, on a point set of $\boldsymbol{S}$ it is possible to find the worst point by each criterion of $k = \overline{1, K}$, i.e., $X^0_k \in \mathbf{R}^N$ - the worst point and $f^0_k \equiv f_k(X^0_k)$ is the value of the criterion in this point, VPMP (1.1.1)-(1.1.4) is received with the decision made separately for each component of vector criterion, (the problems with $k \in \boldsymbol{K}_1$ are solved at a minimum of $f^0_k \equiv f^{\min}_k$, and with $k \in \boldsymbol{K}_2$ respectively at a maximum of $f^0_k \equiv f^{\max}_k$). From here, it follows that each criterion of $k \in \boldsymbol{K}_1 \subset \boldsymbol{K}$ on an admissible set of $\boldsymbol{S}$ can change from $f^0_k \equiv f^{\min}_k$ to $f^*_k \equiv f^{\max}_k$:

$$f^0_k \leq f_k(X) \leq f^*_k, \; k = \overline{1, K_1}, \tag{1.2.1}$$

and the criteria $k \in K_2 \subset K$ on the admissible set $\boldsymbol{S}$ vary from the maximum value $f^0_k = f^{\max}_{\mathrm{k}}$, $k = \overline{1, K_2}$ to f^*_k, $k = \overline{1, K_2}$:

$$f^0_k \geq f_k(X) \geq f^*_k, \; k = \overline{1, K_2}. \tag{1.2.2}$$

In (1.2.1), (1.2.2) f^0_k, $k = \overline{1, K}$ is what we call the worst part of the k-th criterion.

2) We recall the definition of the set of Pareto optimal points.

Definition (condition of optimality of a point on Pareto):

In VPMP, the point of $X^o \in \boldsymbol{S}$ is Pareto-optimal if it is admissible and there is no other point of $X' \in \boldsymbol{S}$ for which:

$f(X') \geq f(X^0_k)$, $k=\overline{1, K_1}$, $f_k(X') \leq f_k(X^o)$, $k=\overline{1, K_2}$

and at least for one criterion is carried out with strict inequality.

The point set of $\boldsymbol{S^o}$ for which the condition of optimality of a point on Pareto is satisfied is called a point set, Pareto-optimal, by $S^o \subset S$. This is also called a set of "not-improved points".

The theorem (concerning the existence of Pareto-optimal points):

In VPMP (1.1.1)-(1.1.4), if the set of admissible points of $\boldsymbol{S}$ is not empty and represents a compact, and vector criterion (1.1.1) - concave functions, and vector criterion (1.1.2) – convex functions, then a point set of $\boldsymbol{S^o}$, Pareto-optimal, is not empty: $\boldsymbol{S^o} \neq \emptyset$, $\boldsymbol{S^o} \subset \boldsymbol{S}$. The proof is in [24].

Point set, being Pareto-optimal, are somewhat between optimum points which are received as a result of the solution to VPMP separately by each criterion. For example, in two criteria of VPMP, as shown in Fig. 1.1, $\boldsymbol{S^o}$ represents some curve $X^*_1\ X^*_2$.

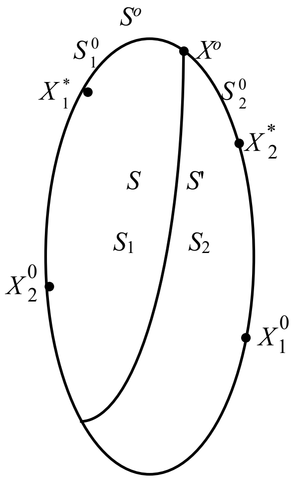

Fig. 1.1. Geometrical interpretation of an example of the solution to two-criteria VPMP:
X^*_1 , X^*_2 - optimum points in the solution to VPMP in the first and second criteria; X^0_1, X^0_2 - the worst points, respectively;
X^o - the decision at equivalent criteria;
$\boldsymbol{S}$ - an admissible point-set in VPMP;
$\boldsymbol{S^o}$ - points, Pareto-optimal (subset);
$\boldsymbol{S_1}$, $\boldsymbol{S_2}$ - admissible points (subsets) which are the priority for the first and second criterion respectively.

Let's note that such path limits (sides) of change for any criterion can be found on a Pareto set: for criterion to maximize $q \in \boldsymbol{K}_1$ its minimum size is defined by viewing all optimum points on a set of maximizing criteria for the first type:

$f_q^{min} = \min_{k \in K_1} f_q(X_k^*),\ q = \overline{1, K_1},$

where X_k^* - an optimum point which is received in the solution to VPMP, separately to the k-th criterion; for minimization criteria for the second type there is the maximal size, which is defined by viewing all optimum points on the corresponding set of maximizing criteria:

$f_q^{max} = \max_{k \in K_2} f_q(X_k^*),\ q = \overline{K_1 + 1, K}.$

Thus, the values of criterion $f_q(X)$ received on a Pareto set lie in borders:

$$f_q^{\min} \le f_q(X) \le f_q^*,\ q = \overline{1, K_1}, \quad (1.2.3)$$

$$f_q^{\max} \ge f_q(X) \ge f_q^*,\ q = \overline{K_1 + 1, K}. \quad (1.2.4)$$

Let's notice that $f_q^{\min}$ can be the very worst decision and f_q^0, i.e. $f_q^0 \le f_q^{\min}$, $\forall q \in \boldsymbol{K}_1, f_q^{\max} \le f_q^0$, $\forall q \in \boldsymbol{K}_2$ is similar.

3) There is a natural question about whether the point can, if it's Pareto-optimal, be the solution to VPMP (1.1.1)-(1.1.4).

The answer is no.

Generally speaking, in VPMP (1.1.1)-(1.1.4), the point set of $\boldsymbol{S}^o$ is commensurable or can even coincide with a set of admissible points of S. Below are two examples illustrating this premise.

Example 1.1. Let's consider a vector problem with two linear criteria.

$$\begin{aligned} &max\ F(X) = \{max\ f_1(X) \equiv 2x_1 + x_2,\\ &\qquad max\ f_2(X) \equiv x_1 + 2{*}x_2\}, \quad (1.2.5)\\ &\qquad x_1 + x_2 = 1,\ x_1, x_2 \ge 0. \end{aligned}$$

The Pareto set of $\boldsymbol{S}^o$ and the set of admissible points of $\boldsymbol{S}$ are equal among themselves and represent a point set, lying on a straight line $X_1^* X_2^*$ (see Fig. 1.2, a) with coordinates:

$X_1^* = \{x_1^* = 1, x_2^* = 0\}, X_2^* = \{x_1^* = 0, x_2^* = 1\}.$

The points of X_1^* and X_2^* are the solution to VPMP (1.2.5), separately, with the corresponding criterion. Moving forward, we will note that the solution to VPMP (1.2.5), on the basis of criteria normalization and the principle of the maximine (a guaranteed result), with a condition of an equivalence of criteria, only has the following appearance:

$\lambda^o = 0{,}5,\ X^o = \{x_1 = 0{,}5,\ x_2 = 0{,}5\}.$

Example 1.2. Let's consider a vector problem with three linear criteria:

$opt\ F(X) = \{max\ F_1(X) = \{max\ f_1(X) \equiv 2x_1 + x_2,$

$\qquad max\ f_2(X) \equiv x_1 + 2{*}x_2\},$

$$min\ F_2(X) = \{min\ f_3(X) \equiv x_1 + x_2\}, \quad (1.2.6)$$
$$x_1 + x_2 \leq 1,\ x_1,\ x_2 \geq 0\ .$$

Sets of $\boldsymbol{S^o}$ and $\boldsymbol{S}$ are equal among themselves and represent point sets, lying on the plane $X_1^* X_2^* X_3^*$ (see Fig. 1.2, b) with coordinates:

$X_1^* = \{x_1^* = 1,\ x_2^* = 0\}$, $X_2^* = \{x_1^* = 0,\ x_2^* = 1\}$, $X_3^* = \{x_1^* = 0,\ x_2^* = 0\}$.

Points of X_1^* *and* X_2^*, to X_3^* submit the solution to VPMP (1.2.6) for each criterion respectively.

The result of the solution to a vector task (1.2.6) at equivalent criteria:

$\lambda^o = 0{,}43$, $X^o = \{x_1 = 0{,}285,\ x_2 = 0{,}285\}$,

$\lambda^o = \lambda_1(X^o) = \lambda_2(X^o) = \lambda_3(X^o) = 0{,}429$.

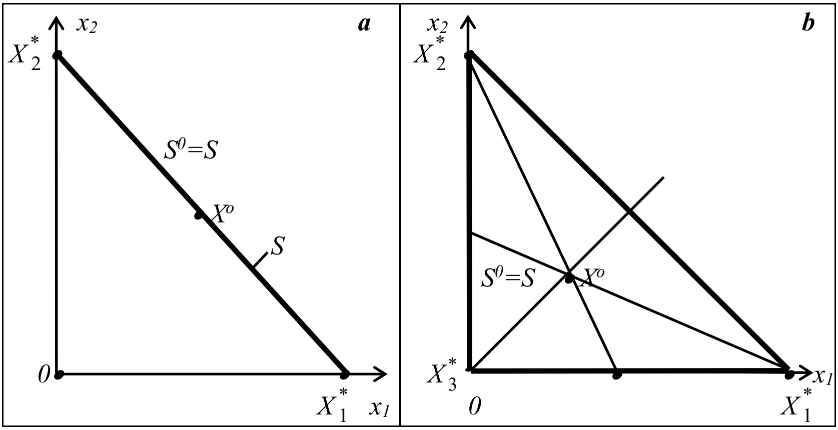

Fig. 1.2. Geometrical interpretation of distribution point-sets in VPMP: a) with two, and b) three criteria.

The given examples show that if Pareto-optimal is found in VZMP (1.2.5) and (1.2.6) points, then only the admissible point is found, and no more. The answer to the question of what this is better than, other than points from a Pareto set, remains open.

Generally, for VPMP (1.1.1)-(1.1.4), there is a problem not only with the choice of a point, Pareto-optimal $X^o \in S$, but also with definition, in that a point of $X^o \in S^o \subset S$ is "more optimum" than another point of $X \in S^o$, $X \neq X$, i.e., the choice of the principle of optimality on a Pareto set.

Let's consider the example of VPMP with two criteria and the question of the preferences (or priorities) of the person making the decisions.

Apparently, if the decision-maker considers the first criterion to be the priority, then the greatest priority will be at a point of an optimum of X_1^*; further from X_1^*, the first criterion's priority concerning the second decreases. If the decision-maker considers the second criterion to be the priority, then the greatest priority for it, apparently, will be at a point of X_2^*; with removal from it, the priority of the second criterion concerning the first decreases. At some distance from X_1^* and X_2^* there has to be a compromise point of X^o in which neither the first nor second criterion have priority, i.e., they are equivalent.

Thus, the question of the preferences (or priorities) of the decision-maker demands a more precise definition in terms of the area of priority according to this or that criterion, and the areas where criteria are equivalent.

Generally, we will try to formulate the problem of finding the solution to VPMP (1.1.1)-(1.1.4).

According to us, the problem with finding the solution to VZMP is in the ability to solve three problems:

• First, to allocate any point from a set of $S^o \subset S$ and to show its optimality concerning other points belonging to a Pareto set;

• Secondly, to show by what criterion of $q \in K$ it (the point) is more of a priority than other criteria of $k = \overline{1, K}$ and by how much;

• Thirdly, if there are changes to the limits of the prioritized criterion in the $\boldsymbol{S^o}$ Pareto set, and these are known (and it is easy to state these in ratios (1.2.3), (1.2.4)), then under the given numerical value of the criterion, to be able to find a point at which the mistake does not exceed the given.

The efforts of most vector-optimization researchers have been directed towards solving this problem in general, as well as its separate parts. For the last three decades, a large number of articles and monographs have been devoted to methods for solving vector (multi-criteria) tasks. These have detailed the theoretical research and methods in the following ways:

1. VPMP solution-methods based on the folding of criteria;
2. VPMP solution-methods using criteria restrictions;
3. Methods of target-programming;
4. Methods based on searching for a compromise solution;
5. Methods based on human-machine procedures for decision-making.

Research and analysis of these methods is presented in Chapter Four. Analysis is carried out by comparing the results of the solution to the test example using these methods, to a method based on the normalization of criteria and the principle of a guaranteed result [11, 14, 17], which is the cornerstone of this book.

CHAPTER 2

THE THEORETICAL BASES OF VECTOR OPTIMIZATION

This chapter presents the basic concepts and definition used in the creation of methods to solve the problems of vector optimization; the principles of optimality in solving vector problems; the theoretical results characterizing the formulated principles of optimality in vector optimization problem-solving; and conclusions on the theoretical bases and methods of vector optimization.

2.1. The basic concepts and definitions of vector optimization

To develop the principles of optimality and methods for solving the problems of vector optimization, we will look at the following concepts:

- the relative assessment;
- the relative deviation;
- the relative level;
- a criterion prioritized in VPMP over other criteria;
- a criterion prioritized vector in VZMP over other criteria;
- the given vector of a criterion prioritized in VPMP over other criteria [11, 27].

From this, we will derive a number of definitions that allow us to formulate the principles of optimality in solving vector-optimization problems.

2.1.1. The normalization of criteria in vector problems.

Normalizing criteria (mathematical operation: the shift plus rationing) presents a unique display of the function $f_k(X)$, $k = \overline{1,K}$, in a one-dimensional space of $\boldsymbol{R}^1$(the function $f_k(X)$, $\forall k \in \boldsymbol{K}$ represents a function of transformation from a N-dimensional Euclidean space of $\boldsymbol{R}^N$ in $\boldsymbol{R}^1$). To normalize criteria in vector problems, linear transformations will be used:

$$f_k(X) = a_k f_k^{'}(X) + c_k\ \forall k\in\boldsymbol{K}, \tag{2.1.1}$$

$$\text{or } f_k(X) = (f_k^{'}(X) + c_k)/a_k\ \forall k\in\boldsymbol{K}, \tag{2.1.2}$$

where $f_k^{'}(X)$, $k = \overline{1,K}$ - aged (before normalization) value of criterion; $f_k(X)$, $k = \overline{1,K}$ - the normalized value, a_k, c_k - constants.

Normalizing criteria (2.1.2) in an optimizing problem does not influence the result of the decision, i.e., the point of an optimum of X_k^*, $k = \overline{1,K}$ is the same for non-normalized and normalized problems.

There are two basic requirements in the mathematical operation known as "the normalization of criteria" when applied to vector problems in mathematical programming:

a) the criteria should be measured in the same units;

b) at the optimum points X_k^*, $k = \overline{1,K}$ all criteria must have the same values (e.g., equal to 1 or 100%). These requirements are reflected in the following definitions.

2.1.2. The relative evaluation of criterion.

Definition 1. In a mathematical-programming vector problem (1.1.1)-(1.1.4) $\lambda_k(X)$ is the relative estimate which represents the normalized criterion: $f_k(X)$, $k\in K$ in $X\in S$ point, with a normalization of the following type:

$$\lambda_k(X) = \frac{f_k(X) - f_k^o}{f_k^* - f_k^o},\ \forall k\in K, \tag{2.1.3}$$

where at the point $X\in S$ the value of the k-th criterion is $f_k(X)$; f_k^* - the criterion value k-th at a point of optimum of $X\in\boldsymbol{S}$ is received in solving a vector problem (1.1.1)-(1.1.4) separately to the k-th criterion; f_k^o - the worst size of the k-th criterion in an admissible set of $\boldsymbol{S}$ in a vector problem (1.1.1)-(1.1.4).

It follows from normalization (2.1.3) that any relative estimate of a point of $X\in\boldsymbol{S}$ on the k-th criterion conforms to both requirements imposed on normalization: first, that criteria are measured in the relative units; secondly, that the relative estimate of $\lambda_k(X)$ $\forall k \in\boldsymbol{K}$ in an admissible set changes from zero at a point of X_k^o, $\forall k \in \boldsymbol{K}$ $\lim\limits_{X \to X_k^o} \lambda_k(X)=0$ to the unit at an optimum point X_k^*, $\forall k \in \boldsymbol{K}$ $\lim\limits_{X \to X_k^*} \lambda_k(X)=1$:

$$\forall k \in \boldsymbol{K} \;\; 0 \le \lambda_k(X) \le 1, \; X \in S. \qquad (2.1.4)$$

2.1.3. The relative deviation.

Definition 1a. The relative deviation, $\overline{\lambda}_k(X)$, $k=\overline{1,K}$, is also the normalized criterion of $k \in K$ at a point of $X \in S$ VPMP (1.1.1)-(1.1.4), but with normalization of the type:

$$\overline{\lambda}_k(X) = (f_k^* - f_k(X))/(f_k^* - f_k^o), \; \forall k \in K, \qquad (2.1.5)$$

where $f_k(X)$, f_k^*, f_k^o - the values defining the k-th criterion are also described above.

From (2.1.3) it follows that any relative deviation $\overline{\lambda}_k(X)$ at a point of $X \in S$ on the k-th criterion also conforms to both requirements of normalization:

$$\forall k \in K \lim_{x \to x_k^*} \overline{\lambda}_k(X) = 0; \; \forall k \in \boldsymbol{K} \lim_{X \to X_k^o} \overline{\lambda}_k(X) = 1. \qquad (2.1.6)$$

Between $\lambda_k(X), \overline{\lambda}_k(X)$, $k=\overline{1,K}$, $\forall X \in \boldsymbol{S}$ there exists a one-to-one association:

$$\lambda_k(X) = 1 - \overline{\lambda_k}(X), \; k=\overline{1,K}, \; \forall X \in S. \qquad (2.1.7)$$

The relative estimates and deviations within the corresponding types of normalized criteria, taking into account types of VPMP and types of restrictions, are given in Table 2.1 where designated: $\lambda(X)$, $k = \overline{1,K}$ - the relative estimates of the k-th criterion; $\overline{\lambda}_k(X)$, $k = \overline{1,K}$ - the relative deviation from $X \in \boldsymbol{S}$ optimum by criterion.

Table 2.1: The normalization of criteria in problems of vector optimization.

Type of VPMP	Type of restrictions	Type of normalized criteria	Limits of normalization criterion
The Homogeneous criteria of maximizing (1.1.1),(1.1.3)-(1.1.4)	$0 \le f_k(X) \le f_k^*$	$\lambda_k(X) = f_k(X)/f_k^*$	$0 \le \lambda_k(X) \le 1$
		$\bar{\lambda}_k(X) = (f_k^* - f_k(X) / f_k^*$	$1 \ge \bar{\lambda}_k(X) \ge 0$
	$f_k^{\min} \le f_k(X) \le f_k^*$	$\lambda_k(X)=(f_k(X)-f_k^{\min})/(f_k^*-f_k^{\min})$	$0 \le \lambda_k(X) \le 1$
		$\bar{\lambda}_k(X)=(f_k^*-f_k(X)/(f_k^*-f_k^{\min})$	$1 \ge \bar{\lambda}_k(X) \ge 0$
The homogeneous criteria of minimization (1.1.2)-(1.1.4)	$\infty \succ f_k(x) \ge f_k^*$	$\lambda_k(X) = f_k(X)/f_k^*$	$1 \le \lambda_k(X) < \infty$
		$\bar{\lambda}_k(X) = (f_k(X) - f_k^*)/f_k$	$0 \le \bar{\lambda}_k(X) < \infty$
	$f_k^{\max} \ge f_k(X) \ge f_k^*$	$\lambda_k(X) = (f_k(X) - f_k^{\min})$	$0 \le \lambda_k(X) \le 1$
		$\bar{\lambda}_k(X)=(f_k^* - f_k(X) /(f_k - f_k^{\min})$	$0 \le \bar{\lambda}_k(X) \ge 0$
Non-uniform criteria (1.1.1)-(1.1.4)	$f_k^{\min} \le f_k(X) \le f_k^*$	$\lambda_k(X)=(f_k(X) - f_k^o)/(f_k^* - f_k^o)$	$0 \le \lambda_k(X) \le 1$
	$f_k^{\max} \ge f_k(X) \ge f_k^*$	$\bar{\lambda}_k(X) =(f_k^* - f_k(X)/(f_k^* - f_k^o)$	$1 \ge \bar{\lambda}_k(X) \ge 0$

2.1.4. The relative level

In an operation that compares the relative estimates or their relative deviations, we introduce an additional numerical characteristic of λ, which we will call the relative level.

Definition 2. The relative *level* λ in VPMP is the lower assessment (bound) of a point of $X \in S$ among all relative estimates of $\lambda_k(X)$, $k = \overline{1, K}$, i.e., λ is an essence-lower, bending around the $\lambda_k(X)$ functions:

$$\forall X \in \boldsymbol{S} \;\; \lambda \le \lambda_k(X),\; k = \overline{1, K}, \tag{2.1.8}$$

and the lower level for the realization of a condition (2.1.8) at an admissible point is defined by the formula:

$$\forall X \in \boldsymbol{S} \;\; \lambda = \min_{k \in K} \lambda_k(X). \tag{2.1.9}$$

Ratios (2.1.8) and (2.1.9) are mutually coherent and serve as a transition from the operation of the definition min (max) to the restrictions, and vice versa.

The introduction of the λ level unites all the criteria of VPMP in one numerical characteristic and "makes over" its particular operations, thereby carrying them out with all criteria measured in the relative units. The λ level functionally depends on the $X \in S$ variable – in changing it, it's possible to also change λ. From here, the rule of searching for an optimal solution can also be formulated.

2.1.5. Prioritizing one criterion over others in VPMP

Definition 3. The criterion of $q \in \boldsymbol{K}$ in VPMP at a point of $X \in \boldsymbol{S}$ is prioritized over other criteria of $k = \overline{1, K}$ if assessment by this criterion is more relative or equal to other relative criteria estimates, i.e.,

$\lambda_q(X) \geq \lambda_k(X),\ k = \overline{1, K}$,

and a rigorous priority, for at least one $t \in \boldsymbol{K}$ criterion,

$\lambda_q(X) > \lambda_t(X),\ t \neq q$,

and for other criteria of $\lambda_q(X) \geq \lambda_k(X),\ k = \overline{1, K}$, $k \neq t \neq q$.

Introduction of definition 3 — a criterion prioritized in VPMP executes a redefinition of an early concept of a priority. In earlier definitions, the intuitive concept about the importance of this criterion was invested in the term "priority", but this "importance" is now defined by a mathematical concept: the more the relative assessment of the q-th criterion over others, the more important it is (it has more of a priority), and is the highest priority at the point of an optimum of X_k^*, $\forall q \in \boldsymbol{K}$.

2.1.6. The vector criterion prioritized over other criteria in VPMP.

If the definition of a prioritized criterion is $q \in \boldsymbol{K}$ in VPMP, it follows that it is possible to reveal a point set of $\boldsymbol{S_q} \subset \boldsymbol{S}$ which is characterized by $\lambda_q(X) \geq \lambda_k(X)\ \ \forall k \neq q\ \ \forall X \in S_q$. But the question of whether the criterion of $q \in \boldsymbol{K}$ in this, or in another point of a set of $\boldsymbol{S_q}$, is more prioritized than the others, remains open.

For clarification of this question, we will look at a coupling coefficient between a couple of the relative estimates of q and k which, in total, represent a vector:

$P^q(X) = \{p_k^q(X) \mid k = \overline{1, K}\}\ \ q \in \boldsymbol{K}\ \ \forall X \in S_q$.

Definition 4. In VPMP, with the q-th criterion prioritized over other criteria of $k = \overline{1, K}$, for $\forall X \in S_q$, the vector of $P^q(X)$, which each component shows in how many times the relative assessment of $\lambda_q(X)$, $q \in \boldsymbol{K}$ is more than other relative estimates of $\lambda_k(X)$, $k = \overline{1, K}$ =, we will call