Recent Advances of the Russian Operations Research Society

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Edited by

Fuad Aleskerov and Alexander Vasin

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INTRODUCTION

The articles in this book present the most interesting new results reported at the IX Moscow International Conference on Operations Research (ORM 2018). The Conference is the largest Russian meeting in this area. Lomonosov Moscow State University and Computational Center of the Russian Academy of Sciences hold it every three years, and leading Russian scientists in Operations research take part in it. This time there were more than 330 participants from Russia, Germany, Italy, Sweden, Spain, Turkey, the USA and other countries. They gave 255 talks on the theoretical problems and current applications of Operations Research. Among the most interesting presentations, the Program Committee selected 18 for the publication in this proceedings' volume.

The papers contain new important results and are of interest for researchers and organizations specialized in OR, Game Theory, System Analysis, Macro- and Micro-economic Modelling, Finance and Actuarial Mathematics. The volume may be used for advanced courses in the specified areas for PhD and Master Students. The proposed methods for optimal decision making will be useful for insurance and auditing companies, banks, producers of the software, and other industries.

ORM 2018 was dedicated to the 100 years anniversary of Professor Yu. B. Germeyer (18.07.1918 - 24.06.1975), the founder of the Russian Operations Research School. Several papers in the volume develop his original approaches to optimization and game-theoretic problems.

Yu. B. Germeyer made an outstanding contribution to the development of Operations Research. He formulated the principle of the maximal guaranteed result for decision making under random and uncertain factors, introduced the concept of hierarchical games and proposed efficient methods for computation of their solutions. His books "Introduction to Operations Research" (1971) and "Non-antagonistic games" (1976) remain basic textbooks for students at leading Russian universities, in particular, Lomonosov MSU and at Moscow Institute of Physics and Technology. In 1974 he organized a laboratory of Operations Research in Computing Center of the Academy of Sciences of the USSR. He was also a founder of Operations Research Department at Faculty of Computational

Mathematics and Cybernetics of MSU in 1970. Since that time, he had also been a head of that Department.

The volume consists of three parts. The first one includes game-theoretic models concerning economic interactions with endogenous formation of utility functions, hierarchical structures, global equilibrium with final-offer arbitration, repeated conflicts with variable discounting. The papers discuss different applications of developed models. In particular, they consider markets for derivatives and analyse stability of international environmental agreements.

Next part is about operations research in economics and finance. The papers consider innovations and their possible opposite impact to the growth of GDP and social welfare, social dynamics determined by collective decisions in a society with an elite, the analysis of the prospects for the development of public-private partnerships in large-scale industry, monopolistic competition under heterogeneous labor, interregional trade and different ways of the Russian banking system development. They also discuss new methods to improve reliability of banks' credit risks.

Third part contains papers on a wide range of optimization problems and their applications. The papers consider continuous optimization problems with infinite numbers of constraints, discrete problems related to computation scheduling, development of the Germeyer's approach to decision-making problems in the presence of random and uncertain uncontrolled factors and to correction of improper linear programming problems, the problem of finding solutions of approximate systems of linear algebraic equations, the problem of interpretation of an infeasible solution for a large-scale problem of optimal production planning.

Fuad Aleskerov and Alexander Vasin

Part I.

Game-Theoretic Models

CHAPTER ONE

INFORMATION AND HIERARCHY

FELIX ERESHKO AND MIKHAIL GORELOV

Abstract

The paper discusses the expediency of decentralizing the organizational system control. We consider the problem of decision-making in the presence of external uncertainty factors. Information about external uncertainty is available to the decision maker (the Center); however, the quantity *l* of information that the Center can timely receive and process is assumed to be limited. A combinatorial approach is used to estimate this quantity. The Center can either choose the controlling strategy on its own or entrust the choice of some controls to its agents. In the latter case, the agents make decisions based on their own utility functions. They are considered to be exogenously set. It is assumed that the decision maker retains the right of the first move. We study two systems of models that differ in the attitude of the decision maker towards uncertainty. One of them uses the principle of maximum guaranteed result. The other assumes that the decision maker is risk neutral, and a probability measure is given on the set of uncertainty factors. We prove that, if the interests of the agents are "poorly coordinated" with those of the Center, the centralized control is always more profitable. And if the interests of the Center and the agents are "well coordinated", then, under large values of *l*, centralization of control is more profitable, and under small values of *l*, the decentralized control is preferable.

Keywords: information theory of hierarchical systems, control decentralization, maximal guaranteed result

1. Introduction

This chapter is devoted to answering the question: why is the management of complex systems carried out on a hierarchical basis? One of the answers to this question gives the information theory of hierarchical systems. In a nutshell, this answer looks like this: Hierarchy arises there and then, where and when, for effective centralized decision-making, it is necessary to process too large amounts of information.

The main provisions of the information theory of hierarchical systems were developed in the 1970s, by Yu. B. Germeyer and N.N. Moiseev (Germeier, Moiseev 1971, Moiseev 1975, Moiseev 1981, Moiseev 1999, Moiseev 2003). But until recently, these ideas were developed on a qualitative, informal level. Last, but not least, this is connected with the concept of 'volume of information', which appears in the thesis formulated in the previous paragraph.

Information aspects were considered in the first papers on game theory (see, for example, Von Neumann & Morgenstern 1953, Zermelo 1913). However, in classical game-theory models, the existence of restrictions on the amount of information was not explicitly taken into account, that is, in fact, models were considered without such limits. Such idealization is permissible at a particular stage in the development of the theory. But to discuss the issue under consideration is not acceptable.

Several attempts have been made to formalize the existence of such restrictions (Aliev & Kononenko 2005, Gorelov 2002, Gorelov 2003, Gorelov 2003', Gorelov 2004, Gorelov 2008, Gorelov 2011). Perhaps the most successful was the method proposed in (Gorelov 2011). It allowed the expanding of the range of issues investigated by methods of game theory. For example, it became possible to take into account the presence of errors in the transmission of information arising due to technical failures (Gorelov 2012, 2015), and as a result of purposeful actions by one of the players (Gorelov 2016). It has become possible to take into account the costs of information processing in the models (Gorelov 2017).

In the context of this chapter, it became possible to build the first quantitative models that allow us to associate the expediency of the emergence of a hierarchy with the volumes of information processed (Gorelov & Ereshko 2019, Gorelov & Ereshko 2020). The models studied in (Gorelov & Ereshko 2019) and (Gorelov & Ereshko 2020) differ in the attitude of the operating party to uncertainty. In (Gorelov & Ereshko 2019), it is assumed to be cautious, and in (Gorelov & Ereshko 2020) – risk-neutral. Below an 'intermediate' case will be considered.

Namely, this chapter assumes that the operating party uses the principle of 'value at risk'. This principle is widely used in economic research (Dempster (ed.) 2002, Jorion 2006), but until recently, it was used only in 'optimization' models. It has appeared very recently in game-theory models (Gorelov 2018, 2019).

With all three variants of the attitude of the operating party to uncertainty, the main qualitative conclusion remains unchanged: the higher efficiency of hierarchical management can be explained by the presence of restrictions on the amount of information processed.

2. Basics of the information theory of hierarchical systems

In this section, we will discuss the main thesis of the information theory of hierarchical systems, on an informal level. The rest of this chapter will be devoted to one of the ways to formalize these ideas.

To begin with, we explain how the term 'hierarchy' is understood in this theory.

"When we use the term 'hierarchical structure' or 'hierarchical organization', it means only that the system is divided into separate subsystems, or units, with independent rights for information processing and decision-making" (Moiseev 1975).

This understanding is not universally accepted. For example, Wikipedia gives the following definition:

"Hierarchy is the order of subordination of lower elements to the highest, organizing them into a tree structure; the principle of management in centralized structures" (https://ru.wikipedia.org/wiki/Hierarchy).

The last definition is perhaps the most common. In some cases, it may be adequate. But when studying decision-making tasks, it seems to us to be too narrow, because it reduces the diversity of interrelationships of the elements of organizational systems to the 'boss – slave' attitude, which impoverishes the model to be built and investigated.

It may seem that the definition of Wikipedia is more constructive since it contains a specific mathematical object – a tree. However, if we analyze the existing decision-making models, we will see that only vertices 'remain' in the tree in them.

Thus, there is no 'working' definition of a hierarchy. Analysis of the studied decision-making models shows that the hierarchical structure, in one way or another, is modeled by the used principle of optimality. Roughly speaking, if Nash equilibria are studied, then we are talking about elements at the same level of the hierarchy; and if the principle of the maximum guaranteed result is applied, then systems such as Center–Agent are considered. Unfortunately, in addition to the hierarchy itself, the principle of optimality usually includes the attitude of the subjects to uncertainty and, perhaps, something else. This introduces additional difficulties.

In principle, the elements of the hierarchical structure should, probably, also include the procedures for the exchange of information, i.e., subsystem awareness (see definition above).

In earlier times, the terms ‘hierarchical game’ and ‘game with a fixed order of moves’ were understood as synonyms. One could look here for a basis for a working definition of hierarchy, but the same problem arises. The ‘order of moves’ is used at the stage of building a model and its interpretation, and it is usually absent in the formal model. And, besides, any game can be considered as a game with a fixed order of moves: either moves are made simultaneously, or in turn. So, in a certain sense, the term ‘game with a fixed order of moves’ is tautological.

Also, of course, static decision-making models are a purely preliminary stage of research into inherently dynamic decision-making processes. And after the transition to dynamics, the very notion of the order of moves becomes necessarily blurred: if decisions are made continuously, it is difficult to speak of some ‘order’.

However, it also contains a particular constructive moment. If we consider static models as a result of dynamic aggregation, then something falls into place. The phenomenon of the hierarchy of specific times in various dynamic processes is known. And if the top level element changes its decisions ‘rarely’, and the bottom level elements change ‘often’, then it turns out that, in general, the lower level elements select their controls when the controls of the top level element are already selected and fixed. With the proper aggregation of such processes, that ‘order of moves’ should arise.

Of course, the task of synthesizing optimal hierarchical systems is of current interest. The lack of a ‘working’ definition is, apparently, the main obstacle in attempting to set such a task; one can’t look for ‘I don't know what’. Perhaps the only way out is to consider various pre-defined options for managing one system, analyze them, and compare the results obtained. Thus, from several methods, one can choose the best. This is the path currently being followed, mainly by research. In particular, this is what we will do next.

We continue with another essential thesis.

> “As soon as the system becomes ‘quite complex’, a hierarchical structure inevitably arises in it. We do not know of any complex systems that do not have such an arrangement.” (Moiseev 1975).

This fact, perhaps, can be considered to be generally accepted. Here we focus on the word ‘complex’, which appears twice in the above quotation.

‘Complexity’ is a particular mathematical concept. True, it refers to constructive, not to set-theory mathematics. Unfortunately, most decision-making models are written in the language of set theory. One can see the reason for this is that the founders of the theory of decision-making, E. Zermelo, E. Borel, and J. Von Neumann, were prominent experts in set theory and actively promoted just such an approach. The title of one of the first works on game theory is characteristic: “On the application of set theory to the theory of a chess game” (1912) (Zermelo 1913).

But the main reason is still not the case. Every model should be ‘simple enough’ to be explored. A non-trivial decision-making model must take into account many factors (the possibilities and interests of the players, their awareness and attitude to uncertainty, etc.), and therefore must be sophisticated. To build constructive decision-making models which, on the one hand, are quite informative, and, on the other hand, are reasonably visible, is not possible so far. Such a task seems to be very important, but also quite difficult.

To date, this problem ‘circumvents’, as follows. The models with some additional structures are considered. Structures are usually described in some language, say, in terms of the abstract structures of N. Bourbaki (Bourbaki 2004). Language constructs are quite constructive objects, and one can speak about their complexity quite definitely.

Roughly speaking, if the set A is represented as a Cartesian product $A = A_1 \times A_2 \times \ldots \times A_n$, then the number of factors can be considered as a measure of the complexity of the set. Here, however, one needs to be very careful: the collection of 100 objects located in the form of a square 10×10 is in some sense simpler then a set of the same 100 objects, but ‘heaped up’.

In decision-making models, it is usually not structures of the Cartesian product type which work, but structures of functional spaces. This is because the players’ awareness in many cases is convenient to simulate using strategy-functions. For example, in the classic game Γ_2 of Yu.B. Germeyer, the set of strategies for one of the players is a set of functions. In the game Γ_3, the set of top-level player strategies is already a set of mappings, each of which is defined on a set of functions (Germeĭer 1986). Thus, a particular ‘hierarchy of difficulties’ arises.

This idea can be used to define the term ‘theory of hierarchical games’ (Gorelov & Kononenko 1999). There are a large number of results, which, in general terms can be described as follow: The ‘complex’ decision-making model is explored. The solution of the corresponding problem is written in terms of another, simpler, model, which is connected with the first one. In this kind of research, there is a unity of results and methods. It gives reason to talk about a unified ‘theory of hierarchical games’.

In this regard, it is worth mentioning one more thing. At a very early stage, game theory and optimization theory developed together. But at the moment they are different sections of mathematics. The difference lies just in the plane under discussion. Consider, for example, the simplest case of an antagonistic game. The main task is to calculate the maximum guaranteed result of one of the players

$$\max_{u\in U}\min_{v\in V} g(u,v).$$

Formally speaking, this is the task of optimizing a function $\varphi(u)=\min_{v\in V} g(u,v)$ on the set *U*, and only the particular structure of this function distinguishes game theory from the theory of optimization.

Let us turn to, perhaps, the central question of the information theory of hierarchical systems: why is hierarchy necessary in control systems?

The most general answer to this question, is that it is these systems that ‘survive’ in the process of evolution, since they demonstrate greater efficiency. In practice, one can observe at least two ways of forming hierarchical systems. Some develop spontaneously (for example, the states). Others are created purposefully (say, when the ‘staff list’ is approved). In the first case, we can talk about survival in an aggressive ‘market’ environment, in the second, about survival in this bureaucratic system.

It is easy to argue in favor of higher efficiency of centralized management. To explain the greater efficiency of systems with decentralized control, the mechanism of increasing the efficiency of control with decentralization in the theory under consideration, is revealed as follows:

> “It may turn out that the fully centralized collection and processing of information are either technically impossible, or lead to a significant delay in decision making, i.e., to making decisions on outdated information. In both cases, this will lead to an increase in uncertainty in decision-making procedures, and, consequently, to a decrease in guaranteed estimates of the effectiveness of the management system.
>
> One of the ways to overcome the difficulties caused by a large amount of information and the complexity of its processing is ‘parallelization’ of its collection and processing procedures. However, the decentralization of information processing inevitably requires a certain level of decentralization of decision-making procedures as well, i.e., the creation of independently functioning subsystems.” (Moiseev 1975).

It is also worth mentioning the price of receiving and processing information.

In the context of what was said above, it can be said that decentralization occurs when practical centralized control algorithms turn out to be too complicated. In some cases, decentralized control algorithms may be more simple, and therefore, more efficient.

These two approaches to explaining the emergence of decentralization are quite close, although they do not entirely coincide.

In connection with the above, let us dwell on another question.

If we consistently use the above reasoning, we will come to the following conclusion: as the technique of transmitting information improves, the level of centralization should increase. In recent years, information technology is developing very rapidly. The centralization of management, at least as fast, is not noticeable. How can this paradox be explained?

One can give an explanation. There are two parallel processes. On the one hand, information technologies are being improved. On the other hand, the controlled systems themselves become more complicated. Moreover, it is widely believed that they are becoming more sophisticated, in particular, due to the strengthening of the links between their elements. And, to a certain extent, the expansion of the set of implemented control procedures compensates for the improvement of information technologies.

One of the central concepts in all the previous arguments was the concept of information. Unfortunately, a sufficiently strict and universal definition of the concept of information does not yet exist. It would seem that this presents an insurmountable barrier to the formalization of all the above thoughts. However, this is not entirely true. There is such a section of mathematics as information theory. And it has proven himself in practice. How did they do without a precise definition? The fact is, that in the theory of information, only the particular task of transmitting information is considered, and therefore, it is possible to carry out formalization. But after all, we are faced with a specific task. Therefore, there is hope that the problem will be managed.

And in this sense, the situation is not so bad.

There is an introduced by N.S. Kukushkin (Kukushkin & Morozov 1984) the concept of quasi-informational expansion, giving a kind of ‘external assessment’ of the concept of ‘information exchange’. It is designed in a purely set-theoretical spirit, and therefore is not quite suitable for the discussed purposes, but in many cases, it is very convenient.

On the other hand, it is possible to offer an ‘internal assessment’ of the same concept, since there are quite adequate ways to describe formally specific ways of exchanging information (for example, in the spirit of N. Howard’s meta-extensions (Howard 1966)). Moreover, this ‘assessment’ may well be made constructive.

However, in the context of the issues discussed, the very concept of ‘information’ is not so significant. One other concept is more important. When it comes to the time for ‘processing’ information, the price of such ‘processing’ requires a quantitative measure of the amount of information. There are problems here, but also, not everything is hopeless.

One of the works of A.N. Kolmogorov is called, “Three approaches to the definition of the concept ‘quantity of information’” (Kolmogorov 1965). The article is quite old, but has not lost its relevance. It is clear that if there are three approaches, then none of them is entirely satisfactory. Besides, these general approaches need to be somehow adapted to specific decision-making tasks.

In recent years, several meaningful models of this kind have been constructed, the results of the study of which can be interpreted well. In particular, examples are showing how the complexity of a control system affects the amount of information needed to manage this system effectively.

And one more thesis.

> “As soon as a subsystem receives the right to make decisions, it becomes an independent organism, i.e., inevitably acquires its own goals, in the general case non-identical to the interests of other subsystems and the system as a whole” (Moiseev 1975).

The process of forming goals is objective. Thus, even if some subsystem is artificially created by, say, an operating party, it cannot be assumed that the operating party has the right to set the goals of this subsystem. Of course, it will have the opportunity, by choosing its controls, to influence the value of the subsystem payoff, but the payoff function itself is a kind of exogenous element.

> “The question of the emergence of objective goals among social groups and systems is very complicated. In the final account, he is somehow connected with homeostasis. But all these ties are very mediated, refracted through the prism of the traditions of the social infrastructure, and it is far from easy to trace them” (Moiseev 1981).

Thus, the question of the formation of goals seems is beyond mathematics. However, it is worth noting two results.

The first of them belongs to I.G. Pospelov (Pospelov 1986). Its essence is as follows: If a trader seeks to preserve homeostasis, that is, minimize the probability of his devastation, then he must maximize his profit. This result was obtained, of course, on one simplified model, but it gives hope that, similarly, it will be possible to describe the process of forming goals in some other situation.

Thus, it can be assumed that the criterion of profit widely used in economic research can be explained in evolutionary terms. This, to some extent, describes the survival of the subject in a market environment.

To describe the subject's survival in the bureaucratic system N.N. Moiseev proposed another criterion (Moiseev 1975).

One can describe it as follows: For the 'managerial' element of the hierarchical system, its head assigns a system of indicators P_1, P_2,..., P_m, by which he will evaluate his activity. The same head brings the desired values $P_1^*, P_2^*, ..., P_m^*$ of these indicators to the element. Moiseyev expresses the hypothesis that, by striving for homeostasis, the element will maximize the function $\min\limits_{i=1,2,...,m} \dfrac{P_i}{P_i^*}$.

Models of this kind have been little studied, but they are of considerable interest.

3. Controlled object

Let us proceed to the construction of a mathematical model that formalizes the statements made in the previous section.

Consider the following controlled system model. The operating party can, at its discretion, choose any control w from the set W. In addition to this choice, a specific indefinite factor α from set A, the value of which is not controlled by the operating party, affects the result of the control. The control efficiency is estimated by the value $g(w,\alpha)$ of the function $g: W \times A \to \mathbb{R}$ (as usual $\mathbb{R}$ is the set of real numbers).

We assume that a probabilistic measure $\wp$ is given on the set A, and this measure is known to the operating party. Further, theorems like the law of large numbers will not be used. Therefore, the probability given by the measure $\wp$ can be considered as subjective. It will describe the attitude of the operating party to risk.

Let us take another assumption reflecting the representation of the 'technological structuredness' of the controlled system under consideration. We assume that the set W can be represented in the form of a Cartesian product $W = U \times V^1 \times ... \times V^n$. Then, every element $w \in W$ can be written in the form $w = (u, v^1, ..., v^n)$, where $u \in U$, $v^i \in V^i$, $i = 1, ..., n$. This form of recording, where it is convenient, will be used without additional reservations.

We make the following standard assumptions. We assume that the sets U, V^i, $i = 1, ..., n$ and A are endowed with topologies in which these sets are compact. The function g will be considered continuous in the topology of the Cartesian product $U \times V^1 \times ... \times V^n \times A$. The measure $\wp$ will be regarded as Borel.

Comment. Probably, these assumptions can be weakened without losing all the results obtained below. However, this leads to the need for more accurate, and, as a result, more extended considerations. Since it is not very clear whether there can be interpretations of this model, in which these assumptions will be restrictive, we will not go into these technical details yet.

Topologies on sets $u \in U$, $v^i \in V^i$, $i = 1, ..., n$ induce topology on their product $W = U \times V^1 \times ... \times V^n$. In the future, when it comes to topology on the set W we will keep in mind precisely the topology of the product.

According to Tikhonov's theorem (see (Engelking 1989)), the set W will be compact.

4. Model of centralized control

Suppose that the operating party has the opportunity to receive information about the realized value of the indefinite factor, but the amount of information that it is capable of receiving, and timely about-work, is limited. Namely, we will assume that the operating party can use l bit of information, and there are no other restrictions on the use of information.

This is formalized as follows. Let's introduce the notation. After this, we denote by $\Phi(X, Y)$ the family of all functions that map a set X to a set Y.

The assumption made means that all information on the uncertain factor available to the operating party can be encoded with words $s = (s_1, ..., s_l)$ of zeros and units of length l. A set $\{0,1\}^l$ (Cartesian power of a set $\{0,1\}$) is denoted by a letter S. Since the operating party has no restrictions on access to information on the uncertain factor, the choice of the "coding method" $P : A \to S$ is its prerogative. Besides, depending on the information received $s \in S$, the operating party has the right to choose any control $w \in W$. That is, it can select a function $w_* : S \to W$. If the operating party fixes the encoding method $P \in \Phi(A,S)$ and control selection rule $w_* \in \Phi(S,W)$ and the value of the uncertain factor $\alpha \in A$ is realized, then the operating party will receive a message $P(\alpha)$, select the control $w_*\left(P(\alpha)\right)$ and its payoff will be $g\left(w_*\left(P(\alpha)\right),\alpha\right)$.

Thus, we have described the set $\Phi(S,W)\times\Phi(A,S)$ of strategies (w_*, P) of the operating party. To complete the model, it is also necessary to express its attitude to uncertainty. We do this as follows.

Fix the number $\xi \in [0,1]$. We will assume that the operating party is ready to exclude from consideration the very unfavorable values of an uncertain factor, the total probability of which is less than $1 - \xi$, and is otherwise cautious. Thus, the operating party chooses the set $B \subset A$ and focuses on the result:

$$\inf_{\alpha\in B} g\left(w_*\left(P(\alpha)\right),\alpha\right).$$

Since the set B can be chosen quite arbitrarily, when evaluating the strategy (w_*, P), the operating side is oriented by result:

$$\sup_{B\in\Sigma_\xi}\inf_{\alpha\in B} g\left(w_*\left(P(\alpha)\right),\alpha\right),$$

where Σ_ξ is the class of all measurable subsets B of the set A for which $\wp(B) \ge \xi$. And then, the best result that the operating party can obtain is:

$$R_0(l,\xi) = \sup_{(w_*,P)\in\Phi(S,W)\times\Phi(A,S)} \sup_{B\in\Sigma_\xi}\inf_{\alpha\in B} g\left(w_*\left(P(\alpha)\right),\alpha\right).$$

This formula contains a supremum in the functional space $\Phi(S,W)\times\Phi(A,S)$ and a supremum in the "complex" set Σ_ξ. Therefore, the calculation of $R_0(l,\xi)$ directly by definition, is a complex variational problem. Let us simplify it.

Of course:

$$R_0(l,\xi) = \sup_{B\in\Sigma_\xi} \sup_{(w_*,P)\in\Phi(S,W)\times\Phi(A,S)} \inf_{\alpha\in B} g\left(w_*\left(P(\alpha)\right),\alpha\right)$$

and

$$R_0(l,\xi) = \sup_{B\in\Sigma_\xi} \sup_{w_*\in\Phi(S,W)} \sup_{P\in\Phi(A,S)} \inf_{\alpha\in B} g\left(w_*\left(P(\alpha)\right),\alpha\right).$$

Now we use the following well-known statement: if $f : X \times Y \to \mathbb{R}$, then

$$\sup_{\varphi \in \Phi(Y,X)} \inf_{y \in Y} f(\varphi(y), y) = \inf_{y \in Y} \sup_{x \in X} f(x, y)$$

(see, for example, (Moulin 1981)).

Applying this statement, we get

$$R_0(l,\xi) = \sup_{B \in \Sigma_\xi} \sup_{w_* \in \Phi(S,W)} \inf_{\alpha \in B} \max_{s \in S} g(w_*(s), \alpha)$$

(the set S is finite; therefore the exact upper bound on $s \in S$ is attained).

Fix the function $w_* \in \Phi(S,W)$. It takes $m = 2^l$ different values. Let a set of these values is $\{w_0, w_1, ..., w_{m-1}\}$. The message $s = (s_1, ..., s_l)$ can be considered as a binary record $s_0 ... s_{m-1}$ of a natural number from the set $\{0, 1, ..., m-1\}$. Bearing in mind such identification, it is possible, without limiting generality, to assume that $w_*(s) = w_s$.

With this in mind:

$$R_0(l,\xi) = \sup_{B \in \Sigma_\xi} \sup_{(w_0,...,w_{m-1}) \in W^m} \inf_{\alpha \in B} \max_{s \in S} g(w_s, \alpha).$$

Rewrite this formula as:

$$R_0(l,\xi) = \sup_{(w_0,...,w_{m-1}) \in W^m} \sup_{B \in \Sigma_\xi} \inf_{\alpha \in B} \max_{s \in S} g(w_s, \alpha).$$

Fix an arbitrary number $\gamma < R_0(l,\xi)$. Then there are $(w_0,\dots,w_{m-1}) \in W^m$ and $B \in \Sigma_\xi$, for which

$$\gamma < \inf_{\alpha \in B} \max_{s \in S} g(w_s, \alpha).$$

Consider the set

$$C_0(\gamma; w_0, ..., w_{m-1}) = \left\{\alpha \in A : \max_{s \in S} g(w_s, \alpha) \geq \gamma\right\}.$$

From standard calculus theorems, it follows that for fixed $(w_0,\dots,w_{m-1}) \in W^m$, the value $\max_{s \in S} g(w_s, \alpha)$ continuously depends on α. Therefore, the set $C_0(\gamma; w_0,\dots,w_{m-1})$ is closed. And since the measure $\wp$ is assumed to be Borel, this set is measurable. Since $B \in \Sigma_\xi$, the inequality $\wp(B) \geq \xi$ holds. Also, by construction, $B \subset C_0(w_0,\dots,w_{m-1})$. Hence $\wp(C_0(\gamma; w_0, \dots, w_{m-1})) \geq \xi$ and, therefore, $C_0(\gamma; w_0,\dots,w_{m-1}) \in \Sigma_\xi$.

And conversely, if $\wp(C_0(\gamma; w_0, \dots, w_{m-1})) \geq \xi$, then:

$$\gamma < \min_{\alpha \in C_0(w_0,\dots,w_{m-1})} \max_{s \in S} g(w_s, \alpha),$$

and especially,

$$\gamma < \sup_{B \in \Sigma_\xi} \inf_{\alpha \in B} \max_{s \in S} g(w_s, \alpha).$$

Define the function:

$$\theta(x) = \begin{cases} 1, \text{ if } x \geq 0, \\ 0, \text{ if } x < 0. \end{cases}$$

Then $\wp\left(C_0(\gamma; w_0, \dots, w_{m-1})\right) = \int \theta\left(\max_{s \in S} g(w_s, \alpha) - \gamma\right) \wp(d\alpha)$, and therefore, the last inequality is equivalent to the condition $\int \theta\left(\max_{s \in S} g(w_s, \alpha) - \gamma\right) \wp(d\alpha) \geq \xi$.

Since γ is arbitrary, this implies the following assertion.

Theorem 1. The number $R_0(l,\xi)$ is equal to the least upper bound of the numbers γ, for which one of the conditions is satisfied:

1) the maximum in the following formula is achieved, and the inequality is true

$$\max_{(w_0, \dots, w_{m-1}) \in W^m} \int \theta\left(\max_{s \in S} g(w_s, \alpha) - \gamma\right) \wp(d\alpha) \geq \xi;$$

2) the inequality holds

$$\sup_{(w_0, \dots, w_{m-1}) \in W^m} \int \theta\left(\max_{s \in S} g(w_s, \alpha) - \gamma\right) \wp(d\alpha) > \xi.$$

Comment. The upper bound in the last formula may not be achieved, because we have not made any assumptions about the continuity of the measure $\wp$.

Further, we will need an analog of the considered problem without restrictions on the amount of available information.

We assume that, at the moment of decision making, the operating party knows exactly the realized value of the uncertain factor α. Then, formally, it selects the function $w_\#$ from the set $\Phi(A, W)$. If such a function is chosen, and the value of the uncertain factor α is realized, then the operating party will get the payoff $g(w_\#(\alpha),\alpha)$.

Suppose, as before, that the operating party is ready to neglect events whose total probability is less than $1 - \xi$, and is otherwise cautious. Then the effectiveness of the strategy $w_\#$ will be evaluated by

$$\sup_{B \in \Sigma_\xi} \inf_{\alpha \in B} g(w_\#(\alpha), \alpha),$$

and in general, the operating party can expect to get the result:

$$R_0(\infty, \xi) = \sup_{w_\# \in \Phi(A,W)} \sup_{B \in \Sigma_\xi} \inf_{\alpha \in B} g(w_\#(\alpha), \alpha).$$

Let's transform this formula. Obviously,

$$R_0(\infty,\xi)=\sup_{B\in\Sigma_\xi}\ \sup_{w_\#\in\Phi(A,W)}\ \inf_{\alpha\in B} g\left(w_\#(\alpha),\alpha\right).$$

Further, using the fact given in the proof of Theorem 1, we obtain

$$R_0(\infty,\xi)=\sup_{B\in\Sigma_\xi}\ \inf_{\alpha\in B}\ \max_{w\in W} g(w,\alpha).$$

Fix the number γ and consider the set $C(\gamma)=\left\{\alpha\in A:\max_{w\in W} g(w,\alpha)\geq\gamma\right\}$. In the same way, as was done above, it is established that the condition

$$\sup_{B\in\Sigma_\xi}\ \inf_{\alpha\in B}\ \max_{w\in W} g(w,\alpha)\geq\gamma$$

is equivalent to inequality $\wp\left(C(\gamma)\right)\geq\xi$ or inequality

$$\int\theta\left(\max_{w\in W} g(w,\alpha)-\gamma\right)\wp(d\alpha)\geq\xi\ .$$

This immediately yields the following result:

Theorem 2. The value $R_0(\infty,\xi)$ is equal to the least upper bound of the numbers γ satisfying the condition

$$\int\theta\left(\max_{w\in W} g(w,\alpha)-\gamma\right)\wp(d\alpha)\geq\xi.$$

5. Model of decentralized control

Consider another way to control the same system.

Suppose that the operating party transfers the right to choose the controls v^i to n agents: the agent with the number *i* gets the right to select the control $v^i\in V^i$ $(i=1,...,n)$. The operating party (Center) reserves for himself the choice of control $u\in U$.

As noted above, the emergence of the agent's right to influence the situation inevitably leads to the appearance of his own goals. We will assume that the purpose of the agent is described by the desire to maximize the value of the function $h^i\left(u,v^i,\alpha\right)$. It is essential that this function depends on its own control, the control of the Center, and the uncertain factor, but does not depend on the choices of the other agents.

Comment. In Section 2, it was noted that, from purely mathematical considerations, in principle, it is impossible to understand what goals the specific agents will obtain. Therefore, in this model, goals are set exogenously.

The functions h^i will be considered continuous.

We assume that the Center still has the ability to receive and process l bit of information about the uncertain factor α. Thus, the strategy of the Center is a pair $(u_*, P) \in \Phi(S,U) \times \Phi(A,S)$ of functions $u_* : S \to U$ and $P : A \to S$ (the meaning of these constructions is the same as in the model of the previous section).

Suppose that each of the agents at the time of decision-making has precise information about the uncertain factor.

Suppose the Center reserves the right of the first move, i.e., he first chooses his strategy (u_*, P) and reports it to all agents.

Under these conditions, for any agent *i*, no uncertainty remains. For him, all the controls $v \in V$ are divided into 'reasonable', by choosing whether he will receive a payoff greater or equal to a specific number λ^i, and 'unreasonable', the choice of which promises him payoff less than λ^i (of course, the number λ^i depends on u and α). This principle of behavior is known to the first player, but he is cautious, and therefore counts on the worst result that can be obtained with a 'reasonable' choice of partners. But since the value of α for the Center is not known, for him, this result is a random variable. The attitude of the Center to this uncertainty is as follows: He agrees to exclude from consideration a certain number of 'force majeure' events, the total probability of which is less than a given value $1 - \xi$. For the rest, he orients himself to the worst case for him, and wants to get the maximum guaranteed result. Thus, we arrive at the following definition:

Definition 1. A number γ is called an ξ-guaranteed result of the Center in the model in question, if there is a measurable set $B \subset A$ for which $\wp(B) \geq \xi$, the strategy (u_*, P) and for any $\alpha \in B$ there exist numbers $\lambda^1, \lambda^2, \ldots, \lambda^n$ such that the conditions are met:

1. For any *i*, there exists a control $v_0^i \in V^i$ for which the inequality holds $h^i\left(u_*\left(P(\alpha)\right), v_0^i, \alpha\right) \geq \lambda^i$;
2. For any $\left(v^1, v^2, \ldots, v^n\right) \in V^1 \times V^2 \times \ldots \times V^n$ either $g(u_*(P(\alpha)), v^1, v^2, \ldots, v^n, \alpha) \geq \gamma$, or there exists $i = 1, 2, \ldots, n$ for which $h^i\left(u_*\left(P(\alpha)\right), v^i, \alpha\right) < \lambda^i$.

The least upper bound $R_1(l,\xi)$ of a set of ξ-guaranteed results is called the maximal ξ-guaranteed result of the Center.

One can express the value of $R_1(l,\xi)$ in more traditional terms.

Define the sets

$$BR^i\left(u_*,P,\alpha\right)=$$
$$\left\{v^i\in V^i:h^i\left(u_*\left(P\left(\alpha\right)\right),v^i,\alpha\right)=\max_{v^i\in V^i}h^i\left(u_*\left(P\left(\alpha\right)\right),v^i,\alpha\right)\right\}.$$

By virtue of the compactness of the set V^i and the continuity of the function h^i, such a set is not empty.

The following statement is true:

Theorem 3. The equality holds.

$$R_1(l,\xi)=\sup_{B\in\Sigma_\xi}\ \sup_{(u_*,P)\in\Phi(S,U)\times\Phi(A,S)}\ \inf_{\alpha\in B}\ \min_{v^1\in BR^1(u_*,P,\alpha)}\ldots$$
$$\min_{v^n\in BR^n(u_*,P,\alpha)}g\left(u_*\left(P\left(\alpha\right)\right),v^1,\ldots,v^n,\alpha\right).$$

Proof. Temporarily, we denote the right-hand side of the last equality by R'. First, we prove the inequality $R_1(l,\xi)\le R'$.

Choose an arbitrary $\gamma<R'$. Then there exist a set $B\in\Sigma_\xi$ and a strategy $\left(u_*,P\right)\in\Phi(S,U)\times\Phi(A,S)$ for which:

$$\gamma<\inf_{\alpha\in B}\ \min_{v^1\in BR^1(u_*,P,\alpha)}\ldots\min_{v^n\in BR^n(u_*,P,\alpha)}g(u_*\left(P\left(\alpha\right)\right),v^1,\ldots,v^n,\alpha).$$

Fix an arbitrary $\alpha\in B$ and put $\lambda^i=\max_{v^i\in V^i}h^i\left(u_*\left(P\left(\alpha\right)\right),v^i,\alpha\right)$ (i=1,…,n).

With this choice of λ^i, the inequality $h^i\left(u_*\left(P\left(\alpha\right)\right),v_0^i,\alpha\right)\ge\lambda^i$ holds for any v^i belonging to the non-empty set $BR^i(u_*,P,\alpha)$. Therefore, Item 1 of Definition 1 is fulfilled.

On the other hand, with this choice of λ^i, inequalities, $h^i\left(u_*\left(P\left(\alpha\right)\right),v_0^i,\alpha\right)\ge\lambda^i$ are satisfied only for $v^i\in BR^i(u_*,P,\alpha)$. But for such v^i, by choice of the number γ, the inequality $g(u_*(P(\alpha)),v^1,v^2,\ldots,v^n,\alpha)\ge\gamma$ holds. This means that the second item of Definition 1 is also fulfilled.

Thus γ is the ξ-guaranteed result of the Center in the game in question. Since γ is arbitrary, this implies the inequality $R_1(l,\xi)\le R'$.

Now prove the reverse inequality $R_1(l,\xi)\ge R'$.

Let γ be an arbitrary number greater than R'. Fix arbitrary $B\in\Sigma_\xi$ and $\left(u_*,P\right)\in\Phi(S,U)\times\Phi(A,S)$. Then under the choice γ there exists $\alpha\in B$ for which

$$\gamma > \min_{v^1 \in BR^1(u_*,P,\alpha)} \ldots \min_{v^n \in BR^n(u_*,P,\alpha)} g(u_*\left(P\left(\alpha\right)\right), v^1, \ldots, v^n, \alpha).$$

Choose any point $\left(v_0^1, \ldots, v_0^n\right) \in BR^1\left(u_*, P, \alpha\right) \times \ldots \times BR^n\left(u_*, P, \alpha\right)$ for which

$$g\left(u_*\left(P\left(\alpha\right)\right), v_0^1, \ldots, v_0^n, \alpha\right) = \min_{v^1 \in BR^1(u_*,P,\alpha)} \min_{v^2 \in BR^2(u_*,P,\alpha)} \ldots$$

$$\min_{v^n \in BR^n(u_*,P,\alpha)} g\left(u_*\left(P\left(\alpha\right)\right), v^1, \ldots, v^n, \alpha\right).$$

Then $\gamma > g(u_*\left(P\left(\alpha\right)\right), v_0^1, \ldots, v_0^n, \alpha)$.

If the numbers $\lambda^1, \ldots, \lambda^n$ are such that $\lambda^i \le \max_{v^i \in V^i} h^i\left(u_*\left(P\left(\alpha\right)\right), v^i, \alpha\right)$ for all $i = 1, \ldots, n$, then for all i the inequalities $h^i\left(u_*\left(P\left(\alpha\right)\right), v_0^i, \alpha\right) \ge \lambda^i$ hold, and besides, the inequality $\gamma > g(u_*\left(P\left(\alpha\right)\right), v_0^1, \ldots, v_0^n, \alpha)$ holds. Therefore, Item 2 of Definition 1 is not satisfied with this choice of λ^i.

And if $\lambda^i > \max_{v^i \in V^i} h^i\left(u_*\left(P\left(\alpha\right)\right), v^i, \alpha\right)$ for some i, then for this i it is impossible to fulfill Item 1.

Therefore, the number γ is not a ξ-guaranteed result of the Center in the game in question. By arbitrariness of γ, we obtain from this the inequality $R_1(l,\xi) \ge R'$.

The two proved inequalities together give the desired equality. The theorem is proved.

The proved theorem makes the formulation of the problem of calculating $R_1(l,\xi)$ more familiar, but it does not solve the basic problems since the formula of Theorem 3 contains two 'non-elementary' operations at once: the supremum in $B \in \Sigma_\xi$ and the supremum in $\left(u_*, P\right) \in \Phi(S,U) \times \Phi(A,S)$. Let us deal with their elimination. Here it is more convenient to use Definition 1 directly. But for brevity, we use the language of the predicate calculus.

By definition, a number γ is a ξ-guaranteed result of the Center in the model under consideration, if and only if, the condition

$$\exists B \in \Sigma_\xi \exists (u_*, P) \in \Phi(S,U) \times \Phi(A,S) \forall \alpha \in A \exists (\lambda^1, \lambda^2, ..., \lambda^n):$$
$$\left[\forall i \exists v_0^i \in V^i : h^i\left(u_*\left(P(\alpha)\right), v_0^i, \alpha\right) \geq \lambda^i\right] \&$$
$$\left[\forall \left(v^1, v^2, ..., v^n\right) \in V^1 \times V^2 \times ... \times V^n\right.$$
$$\left.\left(g\left(u_*\left(P(\alpha)\right), v^1, v^2, ..., v^n, \alpha\right) \geq \gamma \vee \exists i : h^i\left(u_*\left(P(\alpha)\right), v^i, \alpha\right) < \lambda^i\right)\right]$$

is true.

This condition can be rewritten as

$$\exists B \in \Sigma_\xi \exists u_* \in \Phi(S,U) \exists P \in \Phi(A,S) \forall \alpha \in A \exists (\lambda^1, \lambda^2, ..., \lambda^n):$$
$$\left[\forall i \exists v_0^i \in V^i : h^i\left(u_*\left(P(\alpha)\right), v_0^i, \alpha\right) \geq \lambda^i\right] \&$$
$$\left[\forall \left(v^1, v^2, ..., v^n\right) \in V^1 \times V^2 \times ... \times V^n\right.$$
$$\left.\left(g\left(u_*\left(P(\alpha)\right), v^1, v^2, ..., v^n, \alpha\right) \geq \gamma \vee \exists i : h^i\left(u_*\left(P(\alpha)\right), v^i, \alpha\right) < \lambda^i\right)\right].$$

Now, one can to rearrange the quantifiers of generality and existence:

$$\exists B \in \Sigma_\xi \exists u_* \in \Phi(S,U) \forall \alpha \in A \exists s \in S \exists (\lambda^1, \lambda^2, ..., \lambda^n):$$
$$\left[\forall i \exists v_0^i \in V^i : h^i\left(u_*(s), v_0^i, \alpha\right) \geq \lambda^i\right] \&$$
$$\left[\forall \left(v^1, v^2, ..., v^n\right) \in V^1 \times V^2 \times ... \times V^n\right.$$
$$\left.\left(g\left(u_*(s), v^1, v^2, ..., v^n, \alpha\right) \geq \gamma \vee \exists i : h^i\left(u_*(s), v^i, \alpha\right) < \lambda^i\right)\right].$$

To get rid of the function u_*, we proceed in the same way as in Section 3. Let's put $u_*(s) = u_s$. Then the previous condition can be rewritten as

$$\exists B \in \Sigma_\xi \exists (u_0, u_1, ..., u_{m-1}) \in U^m \forall \alpha \in A \exists j = 0,1,...,m-1 \exists (\lambda^1, \lambda^2, ..., \lambda^n):$$
$$\left[\forall i \exists v_0^i \in V^i : h^i\left(u_j, v_0^i, \alpha\right) \geq \lambda^i\right] \&$$
$$\left[\forall \left(v^1, v^2, ..., v^n\right) \in V^1 \times V^2 \times ... \times V^n\right.$$
$$\left.\left(g\left(u_j, v^1, v^2, ..., v^n, \alpha\right) \geq \gamma \vee \exists i : h^i\left(u_j, v^i, \alpha\right) < \lambda^i\right)\right].$$

In this formula, there remains one 'non-elementary' operation associated with the quantifier $\exists B \in \Sigma_\xi$. To eliminate it, apply the same idea that was used in the proof of Theorem 1.

Rearrange the existence quantifiers:

$$\exists\left(u_0,u_1,\ldots,u_{m-1}\right)\in U^m \exists B\in\Sigma_\xi \forall\alpha\in A \exists j=0,1,\ldots,m-1 \exists\left(\lambda^1,\lambda^2,\ldots,\lambda^n\right):$$
$$\left[\forall i \exists v_0^i\in V^i : h^i\left(u_j,v_0^i,\alpha\right)\ge\lambda^i\right]\&$$
$$\left[\forall\left(v^1,v^2,\ldots,v^n\right)\in V^1\times V^2\times\ldots\times V^n\right.$$
$$\left.\left(g\left(u_j,v^1,v^2,\ldots,v^n,\alpha\right)\ge\gamma\vee\exists i: h^i\left(u_j,v^i,\alpha\right)<\lambda^i\right)\right].$$

Let $C_1(\gamma;u_0,\ldots,u_{m-1})$ be the set $\alpha\in A$, for which the condition holds

$$\exists j=0,1,\ldots,m-1 \exists\left(\lambda^1,\lambda^2,\ldots,\lambda^n\right):\left[\forall i \exists v_0^i\in V^i : h^i\left(u_j,v_0^i,\alpha\right)\ge\lambda^i\right]\&$$
$$\left[\forall\left(v^1,v^2,\ldots,v^n\right)\in V^1\times V^2\times\ldots\times V^n\right. \tag{1}$$
$$\left.\left(g\left(u_j,v^1,v^2,\ldots,v^n,\alpha\right)\ge\gamma\vee\exists i: h^i\left(u_j,v^i,\alpha\right)<\lambda^i\right)\right].$$

Replacing the quantifiers $\forall$ and $\exists$ with minimum and maximum operators, we can rewrite condition (1) as inequality

$$\max_{j=0,1,\ldots,m-1}\max_{(\lambda^1,\ldots,\lambda^n)}\min\left\{\min_{i=1,\ldots,n}\max_{v_0^i\in V^i}\left(h^i\left(u_i,v_0^i,\alpha\right)-\lambda^i\right),\right.$$
$$\left.\min_{(v^1,\ldots,v^n)\in V^1\times\ldots\times V^n}\max\left[g\left(u_j,v^1,\ldots,v^n,\alpha\right)-\gamma,\lambda^i-h^i\left(u_j,v^i,\alpha\right)\right]\right\}\ge 0.$$

From the standard calculus theorems, it follows that under the assumptions about continuity and compactness, all the maxima and minima in this formula are reached. From the same theorems, it follows that the function

$$\varphi(\alpha)=\max_{j=0,1,\ldots,m-1}\max_{(\lambda^1,\ldots,\lambda^n)}\min\left\{\min_{i=1,\ldots,n}\max_{v_0^i\in V^i}\left(h^i\left(u_i,v_0^i,\alpha\right)-\lambda^i\right),\right.$$
$$\left.\min_{(v^1,\ldots,v^n)\in V^1\times\ldots\times V^n}\max\left[g\left(u_j,v^1,\ldots,v^n,\alpha\right)-\gamma,\lambda^i-h^i\left(u_j,v^i,\alpha\right)\right]\right\}$$

is continuous, and hence the set $C_1(\gamma;u_0,\ldots,u_{m-1})$ is closed. Since the measure $\wp$ is assumed to be Borel, the set $C_1(\gamma; u_0,\ldots,u_{m-1})$ is measurable. Its measure can be represented as

$$\wp\left(C_1(\gamma;u_0,\ldots,u_{m-1})\right)=\int\varphi(\alpha)\wp(d\alpha).$$

The set $C_1(\gamma;u_0,\ldots,u_{m-1})$ is the most extensive set of class Σ_ξ, for which condition (1) is satisfied. From this we get the following result:

Theorem 4. The number $R_1(l,\xi)$ is equal to the least upper bound of the numbers γ, for which one of the conditions is satisfied:

1) the maximum in the following formula is achieved, and the inequality is true

$$\max_{(u_0,\ldots,u_{m-1})\in U^m}\int\varphi(\alpha)\wp(d\alpha)\ge\xi;$$