Synthesis of Adequate Mathematical Descriptions of Physical Processes

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“. . . the abyss opened and was full of stars;
the stars are beyond count, the abyss has no bottom”.
(M. V. Lomonosov).

CONTENTS

ABSTRACT

The problem of mathematically modelling the characteristics of physical processes and their adequacy to real experimental data is considered in this book. Two criteria of adequacy are proposed: quantitative and qualitative. Adequate mathematical descriptions can increase the objectivity of the results of mathematical modelling for future application. These descriptions make it possible to use the results of mathematical modelling to optimize and predict the behavior of physical processes. Interrelations between criteria are also considered.

The need to develop adequate mathematical descriptions of physical processes is considered in more detail for cases where mathematical description is represented by a system of ordinary differential equations. The conditions are obtained to allow us to reduce the problem of adequate mathematical description of quantitative type for the solution of several integral equations of the first type. Methods of obtaining stable solutions are suggested. The domains of application of the obtained solutions are specified. The proposed criteria are easily transferred to mathematical descriptions in algebraic form.

Examples of adequate descriptions of physical processes are given.

Keywords: mathematical modelling; mathematical description; criteria of adequacy; stable adequate mathematical description; regularization; applications.

FOREWORD

It is no exaggeration to state that the number of scientific papers dealing with the use of mathematical simulation (modelling) methods can now be measured in the millions. Even a superficial acquaintance with the modern capabilities of mathematics and mathematical methods can amaze the reader with its versatility, commonality of approaches, depth of research, and forecasting capabilities. Such mathematical methods are currently used in almost all areas of human activity, from mechanics to linguistics. However, with the success of this expansion in their use there is often little justification given for the practical use of the results of mathematical simulation and some of the basic modelling problems are left unsolved. One of these problems is the question of the correspondence of the results of mathematical simulation to real physical processes (the problem of adequacy). Another important issue is the legitimacy of using simulation results to predict the behavior of physical processes.

A great personal experience of studying complex dynamic systems by methods of mathematical modelling has given the author an incentive to study such open questions that have not yet been answered in the scientific literature.

Of course, this topic is in an initial stage of development. Given the diversity of areas that see the application of mathematical modelling methods, one may expect a lot of interesting work for researchers in the future.

The author will be grateful to readers who will send him their critical notes regarding the determination of the adequacy of mathematical description for dynamical systems, as well as reviewers who, on reading the monograph, may have valuable comments.

The author believes that such comments will help improve the content of the book.

The author is grateful to Abramovsky Evgeny R., Professor of the Dnipro National University (Department of Hydromechanics) for much useful discussion and assistance in preparing this work.

The Author

INTRODUCTION

Mathematical modelling (simulation) of physical processes is an important tool for the study of the environment.

Modelling in scientific research has been used since ancient times, and has gradually led to the development of new scientific knowledge, in terms of: technical design; construction and architecture; astronomy; physics; chemistry; biology; and finally, the social sciences. The last century brought the concept of modelling success and recognition in almost all branches of modern science.

Mathematical simulation is a means of studying real objects, processes or systems by replacing real objects with mathematical objects, making their study easier with the help of computers.

A mathematical model offers an approximate representation of real-world objects, processes and systems, expressed in mathematical terms. As far as the researcher is concerned, the significant features of the physical objects studied are retained.

There is no doubt that the process of developing and justifying methods of mathematical modelling will continue and will expand their areas of application in the coming years.

This explains the increased interest in questions of the grounded application of mathematical modelling, including such important questions as how mathematical models are constructed, how they are studied, and how the results of modelling are interpreted. These questions have not yet been fully answered. Of course, we are not talking about the numerous concrete cases of the practical application of mathematical methods, but about studying *the general laws* of their application.

Modelling is closely associated with concepts such as abstraction, analogy, and hypothesis [1]. The model serves as a kind of instrument of knowledge that the researcher places between himself and the object and by which he explores this object of interest.

The need to use the simulation method is determined by the fact that many objects (or problems relating to these objects) are difficult or impossible to investigate directly, or because this research requires a lot of time and money.

The process of studying the properties of an object in a simulation model serves as an independent object of research. One form of such a study is to perform “model” experiments in which the conditions of the model’s functioning are deliberately changed and then the data on its “behavior” is systematized. The final result of this stage involves obtaining a knowledge set for the mathematical model.

The application of models of knowledge transfer, carried out using the original model, helps form a plurality of knowledge about the object. This process of knowledge transfer is carried out according to certain rules. The patterns of knowledge must be adjusted to suit the properties of the original object, which have not been reflected in the model or have been changed in its construction. We can reasonably tolerate any result from the model, if this result is associated with the requisite signs and similarity to the original model. If a certain result of the simulated study is associated with difference from the original model, such a result is transferred to a real object in contradiction of its physical rules.

There are at least two points of view relating to the results of simulation. One of these is based on the fact that during synthesis, modelled connections and laws of interaction that are already known to researchers are used (of course, the unknown relations relevant to a model cannot be used). In this case, the model cannot present any new knowledge about the object. The model can only act as a facility for clarification through which numerical experiments can be carried out, including situations in which the object does not exist.

The second point of view about simulation comes from the fact that the design of the model presents known information (communications, relations) about the elements of the object, but they are in accordance with the specific aspects of a complex system, which can be combined to show a qualitatively new property, one that is not inherent to the individual elements. In this case, the mathematical modelling is able to give new and previously unknown knowledge about the object. The second approach to the results of simulation is more optimistic than the first. It does not limit the ideas of the researcher and does not locate them within some set frames. Such an approach offers the hope of acquiring new knowledge about real processes and deepens the sense of the simulation.

Some definitions and some concepts are given for the convenience of exposition.

A *mathematical model* of a real object presents the mathematical dependencies and connections between the elements of a mathematical model. These elements are chosen on the basis of the interests of the researcher and the ultimate goal in studying the object. Usually, dependencies and relationships have forms of differential equations, integral equations, and algebraic connections etc.

The functions of external influences and external loads that are present in the mathematical model of the object in the form of symbols are called models of *external loads.*

The initial conditions, boundary conditions, and other conditions for the mathematical model will be called *additional conditions*.

The totality of the mathematical model of the object, models of external influences, and additional conditions will be called the *mathematical description* of the object.

The study of the behavior of the mathematical model of an object under the influence of models of external loads and additional conditions will be called *mathematical modelling*, *mathematical simulation* or simply *simulation*.

Mathematical modelling or simulation is not the only source of knowledge about an object. Process modelling is "immersed" in a more general process of cognition. This circumstance is taken into account, not just directly during the construction of the model, but also at the completion of the simulation when the association and generalization of research results, obtained through diverse means of cognition, are made.

The work given here is devoted to consideration of problems of the specified type. The author hopes that the study of these problems will be useful in creating adequate mathematical descriptions of real physical processes.

This work is structured as follows.

In the introduction, general problems of mathematical modelling, the discussion questions, and prospects for further development are set forth.

In the first section, general questions relating to the mathematical modelling of real processes are considered. An analysis of the process of mathematical modelling is executed. The preliminary definitions of an adequate mathematical description are given.

In the second section, four examples of constructing mathematical models of real processes with subsequent analysis of the results of mathematical modelling are considered. Each example presents the methodological features of the synthesis of mathematical models; the features necessary for checking the correctness of the results (adequacy); and the specific features necessary for the further use of the results of mathematical modelling. The aim of presenting these examples is to attempt to systematize the general characteristics of the mathematical modelling process for various physical processes. The full cycle of the mathematical modelling process is considered for concrete examples with the purpose of determining common laws and general principles.

In the third section, the framework of one of two approaches (in the following we will only mean this one approach) for solving equations is presented for the problem of synthesizing an adequate mathematical description. Necessary and sufficient conditions are obtained, for which the solution of the synthesis problem exists. It is shown that, in most practical cases, the problem of synthesizing an adequate mathematical description is an ill-posed problem [2, 3]. It is noted that the problems of synthesis have number of specific features that distinguish them from traditional ill-posed problems [4 - 7].

The next section is devoted to the problem of obtaining stable solutions to the problems of synthesis, which can be used in practice. A suitable method for solving this problem is the method of regularization with allowance for noted singularities [8, 9]. The influence of the choice of a stabilizing function on the solution of the synthesis problem is studied. The influence of an uncontrolled error in the initial conditions on the solution of the synthesis problem is shown.

Mathematical models in algebraic form are widely used [10, 11]. These models have a number of specific features for synthesis, verification of adequacy, and further use. As a concrete example, the problem of synthesizing a mathematical model of the steelmaking process is considered. The advantages of the regularization method are shown.

In conclusion, a number of recommendations are formulated and methodological conclusions drawn.

1.

GENERAL QUESTIONS ON THE MATHEMATICAL SIMULATION OF PHYSICAL PROCESSES

1.1. Mathematical simulation and its role in the study of physical processes

The rapid development of computer technology has enhanced the scope of methods of mathematical modelling in both the physical and the humanitarian spheres [12, 13]. The contemporary situation in this field has been characterized by Shannon [14], who admits that despite the extensive literature on the substantiation and study of accuracy of modelling, these questions still remain almost as difficult as they were at the beginning of their development. By constructing mathematical models of a real system, the most important question relates to the conformity (in some reasonable sense) of the characteristics of the model to the characteristics of the real system, as obtained during experimentation. In other words, this author specifies the importance of the problem of the adequacy of mathematical descriptions for real processes.

We will consider a typical situation that arises during the analysis of new processes or phenomena. It is supposed that a real physical process is being observed and values derived from experimental measurement of some characteristics of this process are given. It is necessary to develop an adequate mathematical model of this process for further use.

The situation may be explained by the use of a simple example, presented in [15], where the modelling of an inclined mechanism is submitted. The equations describing the tilted mechanism were chosen in the form

$$\begin{cases} \dot{x}_1 = x_2 , \\ \dot{x}_2 = k_1 x_1 + k_2 x_2 + E(t), \end{cases} \tag{1.1}$$

where k_1, k_2 are coefficients and $E(t)$ is the step-function.

Subsequently, the parameters of the mathematical model (1.) of the system are selected in order to provide the revelant conditions, where the results of simulation properly coincide with experiment. In other words, an adequate mathematical model has been chosen. Analogical investigations were performed using [16, 17].

There are several steps inherent to the process of simulation in any field:

1. Statement of the problem and its qualitative analysis;
2. Synthesis of mathematical description;
3. Preparation of the initial data;
4. The development, analysis and application of numerical results.

Mathematical models of physical processes can be presented as systems of ordinary differential equations, systems of partial differential equations, algebraic relations, or integral equations, etc.

1.2. The process of constructing mathematical descriptions (2 approaches)

Mathematical simulation is a cyclical process. This means that a first three-step cycle may be followed by a second, and a third etc. As knowledge about the object is expanded and refined, the original model is gradually improved. Defects discovered after the first cycle of simulation, from small errors and limited knowledge of the construction of the object's model, can be corrected in subsequent cycles.

Currently, there are two fundamentally different approaches to solving the problem of synthesizing proper mathematical descriptions [18, 19, 20, 21, 22]:

- for a given mathematical model of the physical process, models of external loads are selected in order to obtain acceptable results of mathematical modelling according to comparison with experimental data [18, 19];
- for models of external loads, which are set *a priori*, the coefficients of the mathematical model are selected to obtain "good" results of mathematical modelling in comparison to the experimental measurements [20, 21, 22]. In some cases, within the framework of this approach, the structure of a mathematical model can also change (approximation in a broad sense).

In the process of selecting a mathematical description of an experiment, a number of open questions arise. The first one may be formulated as follows: is it possible to construct a mathematical description for which the results of mathematical modelling would give the best coincidence with a future experiment? The second question asks: would the results of mathematical modelling coincide with the experimental data if models of external loads, which are constructed on the basis of physical representations, were used in mathematical description?

Similar questions and problems have arisen in a number of other works [23, 24, 25].

In this work, mathematical models of physical processes, which are basically described by the system of ordinary differential equations, are examined [26, 27]. Such idealization of real processes or dynamic systems is widely used in various areas of research and application for the description of control systems [26]; mechanical systems with concentrated parameters [28 - 31]; economic processes [32, 33]; biological processes [34]; and ecological processes [35, 36], etc. Human emotions have been simulated with the help of such systems in some research [37].

It is notable that the authors of works on mathematical modelling rarely concern themselves with the question of the adequacy of constructed mathematical descriptions of processes in terms of real measurements.

The considered situation requires the formation of a uniform approach to this problem, including a common methodological approach, general algorithms, and common criteria of evaluating the degree of adequacy.

The research has a general character and, therefore, it requires the analysis of concrete processes in different areas of practice. The theoretical results will be valid for a wide class of processes.

Suppose that a mathematical description has already been constructed using a particular approach. To use it further for the purpose of predicting the behavior of a physical process, it is important to have confidence that the description under the new conditions will remain proper. According to the definition given above, the checking of the mathematical description can be performed only with the use of experimental measurements, which correspond to the new conditions. But then a mathematical simulation is inappropriate for the purpose of predicting the behavior of a physical process by means of mathematical modelling. Thus, in principle, a forecasting regime cannot verify a mathematical description.

To justify the plausibility (reliability) of forecasting in a new environment we may use the property of inertia of dynamical systems: small changes in the initial data correspond to small changes in the behavioral characteristics of dynamic systems. Therefore, if you use a mathematical description in new conditions, which vary only slightly, then the results of mathematical modelling will only be slightly different from previous results, for which experimental data are already available. In such a case there is no sense in using mathematical modelling.

Similar results can be obtained in a case when mathematical description is robust in terms of small changes to the initial data. For example, if the model of an external load (which is part of a mathematical description) is robust in terms of small changes in the initial data, then under the new conditions, the results of the mathematical simulation will only differ slighlty from future experiments. In such a case, the results of mathematical modelling will not be of scientific interest as they will be almost the same as those of previously conducted mathematical modelling and experimentation.

There is no guarantee that mathematical description will give reliable results in predicting the behavior of the physical process under the new conditions. Let us imagine that with temperature t_1 some mathematical description (function of external load $z_1(t)$ for a given mathematical model) was obtained.

The experimental measurements of the state variables of the mathematical model are carried out for a considerable increase in temperature t_2. Based on this data, a new mathematical description (new function $z_2(t)$ of the external load for a given mathematical model) is constructed.

Further analysis is made of changes in the functions $z_1(t), z_2(t)$ (it is important that a mathematical description remains stable despite small changes in the initial data) when the temperature of the physical process changes from t_1 to t_2. These changes in the function $z(t)$ are extrapolated to the new function $z_3(t)$ for temperature t_3 (when $t_1 < t_2 < t_3$) or interpolated to a new function $z_3(t)$ for temperature t_3 (when $t_1 < t_2 < t_3$). Using a mathematical simulation of the physical process, a new function $z_3(t)$ for a fixed mathematical model can obtain reliable results. It is obvious that such an approach should give reliable information about the continuous change of the parameters of the physical process in the change of temperature from t_1 to t_3. This information can only be obtained from physical experiment and observation. It is necessary to be sure that the physical process does not qualitatively change, for example, if the object does not melt at the new temperature t_3.

The mathematical description of physical processes cannot act as a means of checking reliability before their simulation. Therefore mathematical simulation is often considered to be similar to an artform [14].

This situation greatly reduces the usefulness of mathematical modelling and it sometimes gives negative results. This happens when the results of mathematical simulation contradict experimental measurements.

This situation requires the formation of a unified methodological approach to solving this problem, the creation of algorithms for synthesis, and the formation of criteria for the evaluation of the reliability of the results of mathematical simulation [2, 3, 6, 7, 18].

1.3. The adequacy of mathematical descriptions of natural processes

The practical significance of the results of mathematical modelling or the simulation of physical processes depends on the degree of coincidence between the results of mathematical modelling of the selected mathematical description with the experimental data [2, 6, 7]. This property of the mathematical description of a physical process is usually called *adequacy*. It should be noted that, in many works, the accuracy of the results of mathematical modelling is significantly lower than the accuracy of experimental data.

Similar issues are summarized in shortened form in [40].

General definition of the adequacy of mathematical description of quantitative type. A mathematical description can be considered an *adequate mathematical description (AMD) of quantitative type* of the process being studied if the results of mathematical modelling (simulation) using this description coincide with experimental data with sufficient accuracy [6, 7, 18].

An AMD exists in a close relationship with its relevant experiment, as follows from its definition. Let us consider this relationship in more detail. The general case supposes an infinite set of AMD and under the use of this set the results of mathematical modelling accurately coincide with experimental data corresponding to any experiment on the physical process. In the first approach, this is a set of models of external loads. This follows from the incorrectness of the problem of AMD construction [6, 7].

If the experiment has been conducted with low accuracy, then the AMD set is wider than an AMD set corresponding to the same experiment with higher accuracy.

To select one AMD from the specified set, an additional condition is used. Such an additional condition in the first approach may, for example, be: the condition of smoothness of the model of external loads; the condition of the greatest simplicity of recording the model of external loads; the condition of convenience for further use of the model of external loads on existing equipment; or the condition of minimum energy of the model of external loads (see subsections 4.2, 4.4).

If the accuracy of the experiment changes, then the AMD changes also. At the same time, a more accurate mathematical description of the physical process will correspond to a more accurate experiment under the condition of selection.

Experiments can vary in terms of the accuracy of the measurements. The researcher can choose any option for the construction of the AMD taking into account the possible loss of accuracy.

Certainly, such a definition of adequate mathematical description cannot be considered perfect and requires further specification.

The definition of quantitative type adequacy will be clarified later for some specific types of mathematical models.

If the coincidence of the results of mathematical modelling with experimental data is poor, then further use of this mathematical description is problematic.

It is worth noting that authors of works on mathematical modelling rarely concern themselves with the question of the adequacy of constructed mathematical descriptions of processes in terms of real measurements [37]. Sometimes such adequacy is supported by facts, sometimes authors refer to the results of other authors, and sometimes they do not discuss this issue at all.

The considered situation requires the formulation of a uniform approach to this problem, including a common methodological approach, general algorithms, and common criteria for estimating the degree of adequacy.

Comparing the results of mathematical modelling with experimental data in defining an adequate mathematical description ensures the objectivity of the results of its synthesis. In the literature, this approach is termed an identification method. It involves the estimation of the parameters of an adequate mathematical description based on the results of measurements of the characteristics of the physical process [38].

Mathematical models of physical processes are presented as a systems of ordinary differential equations, systems of partial differential equations, algebraic relations, or integral equations, etc.

Many problems investigated in this work exist in other types of mathematical model of physical processes; for example, mathematical models in the form of partial differential equations [39].

Consider the specified criteria for mathematical descriptions of quantitative type in the form of a system of differential equations.

For the sake of simplicity, we select physical processes with mathematical models in the form of a linear system of ordinary differential equations

$$\dot{x}(t) = Cx(t) + D\,z(t), \qquad (1.2)$$

where C, D are matrices with constant coefficients, which are given approximately; $x = (x_1, x_2, \ldots x_n)^T$ gives the vector-function variables, characterized by the state of the process ($(\cdot)^T$ is a mark of transposition); $z(t) = (z_1(t), z_2(t), \ldots z_m(t))^T$ is the vector-function of the external load; and $x \in X, z \in Z, X, Z$ are normalized functional spaces.

We assume that state variables $x_i(t), 1 \le i \le n$ of system (1.2) correspond to some real characteristics of the process under investigation $\tilde{x}_i(t), 1 \le i \le n$.

By *mathematical description* of the physical process, we mean the set of the system of equations (1.2); the vector of the external load functions $z(t) = (z_1(t), z_2(t), \ldots, z_m(t))^T$; and the initial conditions $x(t_0) = x^0$. In other words, a mathematical description is the collection of mathematical model, model of external influences and initial conditions.

The process of solving the system of differential equations (1.2) under the influence of the selected models of external loads $z(t) = (z_1(t), z_2(t), \ldots, z_m(t))^T$, taking into account the initial conditions $x(t_0) = x^0$, is usually called *mathematical modelling* or *mathematical simulation*.

An adequate mathematical description of a physical process of such type, with respect to all variables $x_1(t), x_2(t), \ldots, x_n(t)$ of quantitative type, will be called a mathematical description for which the results of the mathematical simulation of variables $x_1(t), x_2(t), \ldots, x_n(t)$ coincide well with the results of experimental measurement $\tilde{x}_1(t), \tilde{x}_2(t), \ldots, \tilde{x}_n(t)$ of the characteristic $x_1(t), x_2(t), \ldots, x_n(t)$, with experimental accuracy of the selected functional metric on the segment $t \in [a, b]$.

In practice, the measurement of the characteristics of state variables is limited by only one or two components. We formulate a refined definition of the adequacy of mathematical description for the case of a single variable.

An adequate mathematical description of a physical process of such type with respect to the variable $x_k(t), 1 \le k \le n$ ($ALMD_{x_k}$) of quantitative type is considered a mathematical description for which the results of mathematical simulation of a variable $x_k(t)$ coincide well with the results of experimental measurements $\tilde{x}_k(t)$ of the characteristic $x_k(t)$ with experimental accuracy in the selected functional metric on the segment $[a,b]$.

There are a number of shortcomings in this definition of adequate mathematical description, which do not allow us to uniquely identify an adequate description from all those possible. The coincidence of a part of the state variables with the experimental data does not guarantee the coincidence of the remaining state variables with the experiment. In addition, the deviation of the results of mathematical modelling of experimental data is determined by the properties of the function space X, as well as the type of norms defined in this space.

The value $[a,b]$ of the independent variable is also of great importance. If the size of this interval increases several times, then an adequate mathematical description may lose its properties. Therefore, the above definition of an adequate mathematical description gives only the basic requirements, but does not give a definition of an adequate description.

For the rest of the variables, experimental coincidence is not determined. Adequate mathematical descriptions are similarly determined in the case of several measurements of state variables. The metrics of comparison in this case are determined by the objectives of particular studies.

For nonlinear dynamic systems, the algorithm presented can also be used. For this purpose, it is necessary to limit the area of possible variation of the state variables so that the initial dynamic system can be well approximated by the linear system. Having executed the construction of an adequate mathematical description of such a system, it is then necessary to proceed to the next area of the change of state variables and construct an adequate description by identification of the parameters of the mathematical model (without changing the earlier model of the external load obtained). It is then necessary to proceed to the next area of state variables. After a number of steps, it is possible, by comparison of the results from all stages, to construct a general adequate nonlinear mathematical description of the dynamic system. Publications presenting such a solution are unknown to author.

The criteria of adequate mathematical description of quantitative type offered in this work can be used for other types of mathematical description of physical processes; for example, for mathematical descriptions in the form of partial differential equations. They share many common features.

It can be shown that there are an infinite set of adequate mathematical descriptions for each physical experiment.

In addition, qualitatively different physical processes can have adequate mathematical descriptions for the same experiment.

The author hopes that the criteria of adequacy offered will be useful in the construction of adequate quantitative type mathematical descriptions of other physical processes.

Preliminary definition of adequate mathematical description of qualitative type.

We will consider what properties of mathematical descriptions are valid for further use and what criteria should be selected in the construction of mathematical descriptions.

It will be useful to address the classic research in this area. In [14], the following statement is made: "…imitation modelling is the creation of experimental and applied methodology which is aimed at the prediction of the future behavior of a system."

As such, mathematical descriptions are intended to be used for the forecasting of behavior of real processes. Mathematical modelling should aid in predicting the behavior of real processes under new conditions of operation. For example, it will make it possible to test a more intensive mode of operation of a real machine without risking its destruction. Such tools (mathematical descriptions) allow us to simulate the characteristics of a process in unconventional modes of operation, as well as determining its optimal parameters.

Let us now consider the conditions under which it is possible to further use adequate mathematical descriptions for the "prediction of the future behavior of a system."

Obviously, the structure of a mathematical model, its parameters, and the specific type of external loads are determined by the properties of a real physical process.

If we let the selected structure of the mathematical model of the physical process include the parameters $p = (p_1, p_2, \ldots, p_k)^T$ (for example, the mass of the elements; the stiffness of the elastic elements, etc.), reflecting actual physical characteristics, the structure of the mathematical model also includes dependencies that reflect the real physical patterns and dependencies of the process under study.

For the purposes of further substantiation of mathematical descriptions, it is necessary that there is a one-to-one correspondence between the components of the vector parameters p of mathematical description and the actual physical elements. In addition, it is necessary that the interconnections between the parameters of a mathematical model comply with the physical laws of the process being studied and that the main external loads have been included. This important correspondence will be called the *main compliance* (MC). The execution of the MC makes possible the fulfillment of the criterion of *adequacy of qualitative type* [40]. In other words, a mathematical description of a physical process satisfies the criterion of adequacy of qualitative type if the main correspondence is fulfilled.

An additional requirement for the implementation of the MC is explained, firstly, by the fact that the quantitative agreement of the results of mathematical modelling with a specific experiment is possible for mathematical descriptions of qualitatively different physical processes due to the selection of parameters of mathematical description.

The execution of the MC allows models of external influences to be obtained by the identification method (the first approach in subsection 1.2.), corresponding to real external influences acting on the physical process. At the very least, these models will not contradict the physical meaning. If we return to the example of the synthesis of an adequate mathematical description of the process of mechanical oscillations in the main line of a rolling mill, then it can be argued that the MC has been executed (see subsection 2.1.). By virtue of this, the models of the external influences obtained have a reasonable physical interpretation (and do not contradict the physical meaning). The external load smoothly increases from zero to a steady-state value (see Fig. 2-4).

In addition, if the two criteria of adequacy are fulfilled, there is some certainty that the adequacy of quantitative type for some components of the state variable will be in good agreement with the other state variables of the experimental data.

In the second example, given in section 1, the main correspondence is not fulfilled and, therefore, the application of the results obtained under the new conditions will not be justified.

The algorithm for constructing an adequate mathematical description of qualitative type cannot be formalized, as in the case of the adequacy of a mathematical description of quantitative type. The process of constructing such a description primarily depends on subjective factors, such as the scientific problems of studying the physical process using mathematical modelling methods.

2.

EXAMPLES OF THE SYNTHESIS OF MATHEMATICAL DESCRIPTIONS WITH THE ESTIMATION OF ADEQUACY

Let us consider several examples of mathematical modelling of physical processes that offer different prospects for the further use of results.

2.1. Mathematical simulation of system dynamics

This is the example of a mathematical description that satisfies the criterion of the adequacy of quantitative type for all variables $x_1(t), x_2(t), \ldots, x_n(t)$.

We consider in detail the problem of investigating the dynamics of the main mechanical line of a rolling mill, as in [41, 42]. One variant of its kinematic scheme is presented in Fig. 2-1 (a): 1 - engine, 2 - coupling, 3 - gears, 4 - driveshafts, and 5 - operational barrels.

The four-mass model with weightless elastic connections was chosen as the appropriate mathematical model of the dynamic system of the main mechanical line of the rolling mill [41, 42]. The equations of motion are obtained from Lagrange's equations of the second kind

$$\begin{aligned}
&\ddot{M}_{12} + \omega_{12}^2 M_{12} - \frac{c_{12}}{\vartheta_2} M_{23} - \frac{c_{12}}{\vartheta_2} M_{24} = \frac{c_{12}}{\vartheta_1} M_{eng}(t);\\
&\ddot{M}_{23} + \omega_{23}^2 M_{23} - \frac{c_{23}}{\vartheta_2} M_{12} + \frac{c_{23}}{\vartheta_2} M_{24} = \frac{c_{23}}{\vartheta_3} M_{rol}^U(t); \qquad (2.1)\\
&\ddot{M}_{24} + \omega_{24}^2 M_{2\,4} - \frac{c_{24}}{\vartheta_2} M_{12} + \frac{c_{24}}{\vartheta_4} M_{23} = \frac{c_{24}}{\vartheta_4} M_{rol}^L(t).
\end{aligned}$$

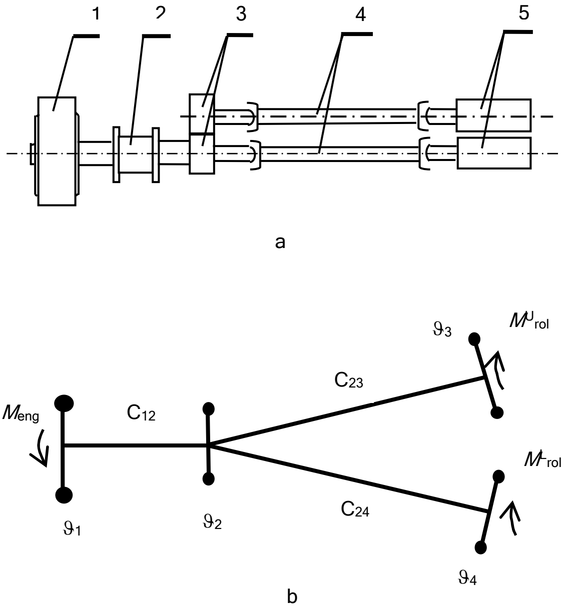

Fig. 2-1. Kinematic scheme of the main mechanical line of a rolling mill.

The following designations were accepted: M_{eng} moment of engine; ϑ_i moments of inertia of the concentrated weights; c_{ik} rigidity of the appropriate elastic connection; M^U_{rol}, M^L_{rol} respective moments of technological resistance on the upper and lower operational barrels; M_{ik} moments of elastic forces, which are applied to the shafts between mass ϑ_i and ϑ_k ; and $\omega^2_{ik} = (\vartheta_i \vartheta_k)^{-1} c_{ik} (\vartheta_i + \vartheta_k)$.

Actually, the constructed mathematical model may or may not correspond to a real process. It is necessary to check the correctness of the constructed mathematical model. For this purpose, experimental data are used. If the results of mathematical modelling coincide with the experimental results (with the accuracy of the measurements), then the mathematical description of the process is considered to adequate to reality in a quantitative sense. In other words, the mathematical description quantitatively corresponds to a real process.

Information related to the real motion of the main mechanical line of the rolling mill was obtained experimentally [18, 43, 44]. Such information is understood to lie in the availability of the functions $M_{12}(t), M_{23}(t), M_{24}(t)$. The values of the functions $M_{12}(t), M_{23}(t), M_{24}(t)$ of the given process are shown in Fig. 2.1.

It is clear that the results of mathematical modelling of the system in (2.1) depend directly on the character of the change in the external loads applied to the operational barrels of the rolling mill and the external impact of the engine M_{eng}, M^{U}_{rol}, M^{L}_{rol}. Sometimes it is possible to pick out the loadings M_{eng}, M^{U}_{rol}, M^{L}_{rol} where the results of mathematical modelling $M_{12}(t), M_{23}(t), M_{24}(t)$ coincide with the values recorded in the experiment in Fig. 2-2.

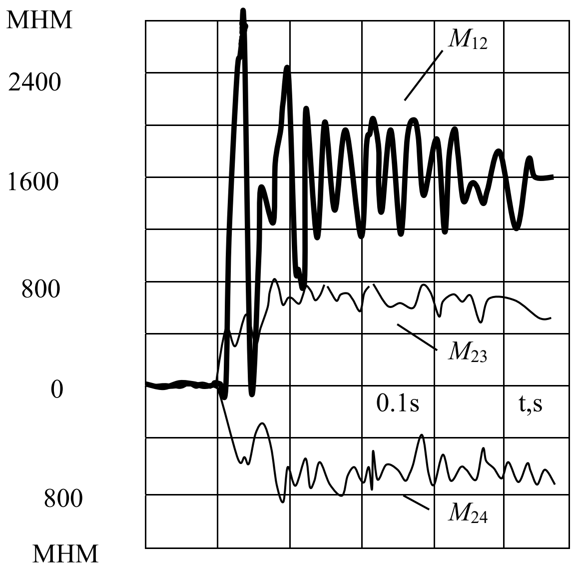

Fig. 2-2. The recorded values of the functions $M_{12}(t), M_{23}(t), M_{24}(t)$.

If such a choice is possible, then the mathematical model (2.1), combined with the loads $M_{eng}, M^{U}_{rol}, M^{L}_{rol}$, will give an adequate mathematical description of the real process. It is important to note that in many papers analyzing the problem of mathematical modelling with the use of a system of differential equations (2.1), the functions $M_{eng}, M^{U}_{rol}, M^{L}_{rol}$ are determined as a mathematical model of the process. Coincidence is understood to relate to the accuracy of the experimental measurements.

According to this approach, it is necessary to construct models of external loads M_{rol}^{U}, M_{rol}^{L} for the system (2.1) where the functions $M_{12}(t), M_{23}(t), M_{24}(t)$ of elastic moments in the links of the model (the solution of the system in (2.1)) coincide with the corresponding experimental functions of the moments of the elastic forces in the links of the main line of the rolling mill (Fig. 2-2).

Consider the construction of an adequate mathematical description within the framework of the first approach. To construct, for example, a model M_{rol}^{u}, which corresponds to the moment of the external load of the upper work roll, we consider the second equation in the system in (2.1). The solution of this equation takes the form

$$M_{23}(t) = M_{23}(0)\cos\omega_{23}t + \dot{M}_{23}(0)\omega_{23}^{-1}\sin\omega_{23}t +$$

$$+\frac{c_{23}}{\vartheta_3\omega_{23}}\int_0^t M_{rol}^{u}(\tau)\sin\omega_{23}(t-\tau)\,d\tau +$$

$$+c_{23}(\vartheta_2\omega_{23})^{-1}\int_0^t [M_{12}(\tau)-M_{24}(\tau)]\sin\omega_{23}(t-\tau)\,d\tau ,$$

or

$$\int_0^t \sin\omega_{23}(t-\tau)M_{rol}^{u}(\tau)\,d\tau = F(t), \tag{2.2}$$

where

$$F(t) = \vartheta_3\,\omega_{23}\,(c_{23})^{-1}\{M_{23}(t) - [M_{23}(0)\cos\omega_{23}t + +\dot{M}_{23}(0)\,(\omega_{23})^{-1}\sin\omega_{23}t\,]\} -$$

$$-\vartheta_3(\vartheta_2)^{-1}\int_0^t [M_{12}(\tau)-M_{24}(\tau)]\sin\omega_{23}(t-\tau)\,d\tau .$$

We assume that function $F(t)$ in (2.2) belongs to the normalized space $L_2[0,T]$ ($[0,T]$ is the period of time at which the function M_{rol}^{u} is studied) and the solution M_{rol}^{u} of equation (2.2) belongs to the normalized functional space $C[0,T]$.

We can rewrite (2.2) in the form

$$A_p z = u_\delta , \tag{2.3}$$

where z is the element looked for; u_δ is the given element to which the functional spaces $C[0,T]$ and $L_2[0,T]$ belong; and A_p is the integral operator. In this case $z = M^u_{rol}(\mathrm{t}), u_\delta = F(t)$.

Since the right-hand side $F(t)$ of the integral equation (2.2) is determined from the experiment, it is natural to assume that instead of the exact right-hand side $u_{ex} = F_{ex}(t)$ of the equation (2.3), some approximation of it is given as $u_\delta = F(t)$

$$\| u_\delta - u_T \|_{L_2[0,T]} = \| F(t) - F_T(t) \|_{L_2[0,T]} \le \delta ,$$

where the value of δ is given.

The set of possible solutions $Q_\delta \subset C[0,T]$ of equation (2.3) consists of elements that correspond to the equation with the given accuracy

$$Q_\delta = \{ z : \| A_p z - u_\delta \|_{L_2[o,T]} \le \delta \} .$$

Each function in the set Q_δ, together with the given mathematical model (2.1), provides an adequate mathematical description of the physical process.

The exact solution of equation (2.3) may not belong to the set of possible solutions of equation (2.3), as the operator A_p in (2.3) does not completely accurately describe the behavior of the real object.

In this case, the problem of identifying a model of the external load in the rolling mill is considered to be the inverse of the synthesis problem. In this case, the value of the error of the operator A_p relative to A_T is considered to be zero. As such, the mathematical model is exactly defined. Similar mathematical models have been used in numerous and varied pieces of research into the dynamics of rolling mills, such as those presented in [45, 46, 47].

In this paper, oscillograms of the moments of the forces of elasticity in the links of the main line of Rolling Mill 1150, obtained in [18, 48], are used. A copy of this oscillogram is shown in Fig. 2-1.

The value 0.48s was chosen as T. The error of the definition of the right-hand side and the error of the operator of the equation (2.3) are then calculated.

If we let the set of possible parameters of a mathematical model be limited by the following values

$$\overline{\vartheta}_2 \le \vartheta_2 \le \hat{\vartheta}_2,\ \overline{\vartheta}_3 \le \vartheta_3 \le \hat{\vartheta}_3,\ \overline{\omega}_{23} \le \omega_{23} \le \hat{\omega}_{23},\ \overline{c}_{23} \le c_{23} \le \hat{c}_{23}, \quad (2.4)$$

where $\omega_{23}^2 = \dfrac{c_{23}(\vartheta_2 + \vartheta_3)}{\vartheta_2 \vartheta_3}$

and, in addition, suppose that

$$\overline{M}_{23}(0) \le M_{23}^{\delta}(0),\ M_{23}^{T}(0) \le \hat{M}_{23}(0);\ \dot{\overline{M}}_{23}(0) \le \dot{M}_{23}^{\delta}(0),\ \dot{M}_{23}^{T}(0) \le \dot{\hat{M}}_{23}(0).$$

then, the estimated error can be obtained

$$\| F_{ex}(t) - F_{\delta}(t) \|_{L_2[0,T]} <$$

$$\le \left\{ \frac{\hat{\omega}_{23}\hat{\vartheta}_2}{\overline{c}_{23}} \delta_{23} + \frac{1}{2\overline{c}_{23}\hat{c}_{23}} \left(\hat{c}_{23}\hat{\omega}_{23}\hat{\vartheta}_2 - \overline{c}_{23}\overline{\omega}_{23}\overline{\vartheta}_3 \right) \left\| M_{23}^{\delta}(t) \right\|_{C[0,T]} + \right.$$

$$+ \frac{\hat{\omega}_{23}\hat{\vartheta}_2}{\overline{c}_{23}} \hat{M}_{23}(0) \left(\hat{\omega}_{23} - \overline{\omega}_{23} \right) T \sqrt{1 - 0.25 (\hat{\omega}_{23} - \overline{\omega}_{23})^2 T^2} +$$

$$+ \dot{\hat{M}}_{23}(0) \frac{\hat{\vartheta}_2}{\overline{c}_{23}} (\hat{\omega}_{23} - \overline{\omega}_{23}) T \sqrt{1 - 0{,}25 (\hat{\omega}_{23} - \overline{\omega}_{23})^2 T^2} +$$

$$+ \frac{T}{2} \left(\frac{\overline{\vartheta}_3}{\hat{\vartheta}_2} + \frac{\hat{\vartheta}_3}{\overline{\vartheta}_2} \right) \left(1 + 0.5 (\hat{\omega}_{23} - \overline{\omega}_{23})^2 T^2 \right) (\delta_{12} + \delta_{24}) +$$

$$+ \frac{T^3}{4} \left(\frac{\overline{\vartheta}_3}{\hat{\vartheta}_2} + \frac{\hat{\vartheta}_3}{\overline{\vartheta}_2} \right) (\hat{\omega}_{23} - \overline{\omega}_{23})^2 \left(\| M_{12} \|_{C[0,T]} + \| M_{24} \|_{C[0,T]} \right) +$$

$$+ T \sqrt{ \frac{1}{4} \left(\frac{\hat{\vartheta}_3}{\overline{\vartheta}_2} - \frac{\overline{\vartheta}_3}{\hat{\vartheta}_2} \right)^2 \left(1 + \frac{(\hat{\omega}_{23} - \overline{\omega}_{23})^4 T^4}{3} \right) + \frac{\hat{\vartheta}_3^2}{\overline{\vartheta}_2^2} (\hat{\omega}_{23} - \overline{\omega}_{23})^2 T^2 } \times$$

$$\left. \times \left[\delta_{12} + \delta_{24} \left(\left\| M_{12}^{\delta} \right\|_{C[0,T]} + \left\| M_{24}^{\delta} \right\|_{C[0,T]} \right) \right] \right\} \sqrt{T} . \quad (2.5)$$