The Dynamic Morse Theory of Control Systems

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Cambridge Scholars Publishing

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This book first published 2020

Cambridge Scholars Publishing

Lady Stephenson Library, Newcastle upon Tyne, NE6 2PA, UK

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

ISBN (10): 1-5275-4507-5
ISBN (13): 978-1-5275-4507-6

For Priscila, Larissa, and Henrique.

“The LORD by wisdom hath founded the earth; by understanding hath he established the heavens. By his knowledge the depths are broken up, and the clouds drop down the dew. My son, let not them depart from thine eyes; keep sound wisdom and discretion: So shall they be life unto thy soul, and grace to thy neck”. Proverbs 3,19-22

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Preface

This book is designed for students or researchers who are interested in dynamic concepts of control systems. It presents classical concepts of control theory integrated with a report about ongoing research on Conley theory for control systems. We assume the reader knows the rudiments of differential equations, real analysis, and topology. For theoretical purposes, we consider control systems on differentiable manifolds. In this setting, we may include many interesting cases of control systems in non-Euclidean spaces, for instance, tori, spheres, and projective spaces. Readers who are not familiar with manifolds may consider Euclidean state space instead. Otherwise, a brief appendix on differentiable manifolds is provided for elementary definitions, notations, and references. An appendix on dynamical systems is provided for references to results from the Conley theory of flows.

This volume expands on the material presented in *A Course on Geometric Control Theory: Transitivity and Minimal Sets* [93]. Its contents includes a presentation of the fundamental theory of control systems, an exposition of elementary concepts of orbits, invariance, periodicity, and a broad discussion on various aspects of transitivity and controllability. The main part deals with attractors and repellers, Morse decompositions, and chain transitivity. Most concepts presented are illustrated by means of comprehensive examples and figures. A list of problems is given in each chapter with the intention of reinforcing the reader's grasp of the material, amplifying and completing proofs, applying theorems, and enabling the reader to discover important facts, examples, and counterexamples. Notes and references are included at the end of each chapter to indicate results not discussed in the text, remarks, references for further reading, and historical notes.

The main contribution of the present work is the improvement of results in dynamic Morse theory for control systems, which are now integrated in this unique volume. The text adds to our knowledge of various dynamical concepts which compose the full ingredients of the central notion of Morse decomposition. Having read this book, the reader will have the opportunity to expand Conley's ideas by means of open questions on Morse decompositions for control system on noncompact manifolds.

Parts of the book may be useful in courses or seminars in mathematics as well as control-theoretic engineering. The material may also be used as a reference source for various topics in control systems, and serve as a basic reference for academic or research projects.

Thanks are given above all to God for life and for the sciences. Special thanks are due to José V. de Souza and Rita C. A. de Souza for moral support; to Prof. Carlos J. Braga Barros, his collaboration and the donation of the Colonius–Kliemann book; and to Prof. Luiz A. B. San Martin, his instructions and our discussions on control theory and transformation semigroups. We acknowledge all our coworkers: Prof. Ronan A. Reis, Prof. Hélio V. M. Tozatti, Prof. Victor H. L. Rocha, and Stephanie A. Raminelli. We also give thanks to João A. N. Cossich for part of the material on differentiable manifolds.

Maringá, Brazil
March 2019

Introduction

The global dynamical behavior of a compact system is described by Morse decomposition. This was stated by Charles C. Conley (1933–1984) in his famous work, “Isolated Invariant Sets and the Morse Index” ([36]), and is currently one of the most important statements in the study of asymptotic behavior of dynamical systems. A Morse decomposition contains essential information about the long-term behavior of a system, since each state converges in forward as well as in backward time to some Morse set. This is due to the attractor-repeller configuration of a Morse decomposition, which provides attractive and repulsive properties for its components. Besides, the attractor-repeller configuration implies an order among the Morse sets such that the flow can be interpreted via its movement from the Morse sets with lower indices toward those with higher ones. This means that, outside a Morse decomposition, every trajectory of the system comes near some Morse set in forward time, and other distinct Morse set in backward time. Consequently, the system does not admit asymptotic transitivity outside a Morse decomposition, or, in other words, any asymptotic transitive set resides in some Morse set. Thus periodicity and recurrence occur only in the components of a Morse decomposition; outside them, however, the system is transient.

In general, the transient part of a dynamical system consists of all states which are not chain recurrent. The chain recurrence is a general type of recurrence based on returning trajectories with jumps. This defines an equivalence relation - the chain transitivity - whose equivalence classes lie in the components of a Morse decomposition. In his report entitled “The gradient structure of a flow” ([37]), Conley proved the *Fundamental Theorem of Dynamical Systems* that any flow on a compact metric space decomposes into the chain recurrent part and the transient or ‘gradient-like’ part. This means that, if each equivalence class of chain transitivity is identified to a point, then the resulting flow has a Lyapunov function that decreases along all trajectories except the fixed points. This was the reason for Conley coining the term, the 'Morse decomposition' of a system. The central result in this direction characterizes a system with finitely many chain equivalence classes as a system admitting the finest Morse decomposition. Conley's ideas about Morse decomposition concentrated in the topological considerations of connecting orbits between the Morse sets. His later studies result in the notion of a 'connection matrix' ([78]), with the main applications being for the theory of shock waves ([38, 39, 40, 41]).

In the Conley global view, the fundamental elements of a dynamical system are the isolated invariant sets. An invariant set is called 'isolated' if it is the maximal invariant set in some neighborhood of itself. This extends the well-known concept of isolated singularity. Attractors, repellers, and Morse sets are the main isolated invariant sets of a system. The 'Morse index' of an isolated invariant set is the homotopy type of an n-dimensional sphere, where n is the dimension of the unstable manifold, the set of points along the flow which are sent away from the isolated invariant set as time moves forward. Conley expanded the ideas about the Morse index by introducing the notions of isolating blocks. An 'isolating block' is a set whose boundary has no internal tangencies to the flow, that is, if the flow is tangential to the boundary of the block, then the trajectories leave the block both forward and backward in time. The 'exit set' is the set of points on the boundary where the flow leaves the block. On the one hand, every isolated invariant set can be surrounded by an isolating block and, on the other hand, every isolating block contains an isolated invariant set in its interior. The 'Conley index' is the homotopy type of the isolating block with the exit set identified to a point. If an isolated invariant set is a non-degenerate rest point for a smooth flow on a compact manifold, then the Conley index and the Morse index coincide.

An isolating block is structurally stable in the sense that it persists under perturbations of the system. Conley believed that this property of isolating blocks meant that they were the only dynamical objects which could be detected in nature and that their properties reflected the important properties of natural systems. This inspires his interest to explore new areas of science with the intention of applying his theories. The Conley index was applied to prove conjectures on the number of fixed points of sympletic maps ([42]). It generalized the Hopf index theorem that predicts the existence of fixed points of a flow inside a planar region in terms of information about its behavior on the boundary. Other applications in the study of dynamics include the existence of periodic orbits in Hamiltonian systems and traveling wave solutions for partial differential equations, structure of global attractors for reaction-diffusion equations and delay differential equations, proof of chaotic behavior in dynamical systems, and bifurcation theory. We use the works [8, 52, 62, 76, 77] to ensure complete information on and references for the Conley index theory.

The proposal of the present book is to reproduce the Morse decomposition part of the Conley theory for control systems. Put more simply, we start with a dynamical system associated with a differential equation $\dot{x} = X_0(x)$ on a manifold (or Euclidean state space), and we then consider a family of differential equations of the form

$$\dot{x} = X_0(x) + \sum_{i=1}^{n} u_i(t) X_i(x)$$

where $X_1, ..., X_n$ are vector fields and $u_1, ..., u_n$ are real-valued functions such that $u(t) = (u_1(t), ..., u_n(t))$ is valued in a given subset $\mathrm{U} \subset \mathbb{R}^n$. The additional terms on the right hand side can be interpreted as control or perturbation. In the control point of view, the set $\mathrm{U} \subset \mathbb{R}^n$ is the given control range and the vector fields $X_1, ..., X_n$ determine the input structure. This family of differential equations defines a control affine system, or in other words, a nominal dynamical system with additional control inputs. A fundamental fact in this definition is that the solutions for the control system do not define a flow on the adjacent state space. We then sought a sense or interpretation for Conley theory in the control system set-up. The motivation for responding to this question was the controllability, the main subject in control theory. Indeed, a set is 'controllable' if its points are pairwise connected by trajectories of the control system. This means that any point in a controllable set has the property of returning trajectories. Thus dynamical concepts, such as periodicity and recurrence could be related to both notions of controllability for control systems and Morse decomposition for dynamical systems. This perception indicated that controllability must relate to some notion of Morse decomposition for control systems.

The book written by Fritz Colonius and Wolfgang Kliemann [34] contained the first attempt to reproduce the Conley theory for control systems. These authors used the strategy of associating a dynamical system with a given control affine system, studying control-theoretic aspects together with the analysis of the associated dynamical system. The so called 'control flow' is an infinite dimensional dynamical system that lives on the product space of admissible inputs and state space. It was extensively used in later works with the purpose of transferring dynamical objects and results from the theory of dynamical systems to the theory of control systems. For instance, the Conley definition of attractor was transferred to control systems by means of the control flow ([35]). Nevertheless, the Colonius–Kliemann definition for chain transitivity of control systems does not use the control flow. Actually, chain transitivity can be defined in general situations of control systems, where the required conditions to constitute the control flow need not be satisfied. In fact, Carlos C. Braga Barros and Luiz A. B. San Martin [20] realized that the chain controllability depends on a family of ideals in the system semigroup associated with the control system. In a straightforward way, they extended the concept of a chain control set to more general semigroup actions, where the chain recurrence came to depend on a family of subsets of the acting semigroup. This idea has been extensively used to generalize dynamical concepts for semigroup actions, which, in particular, composed a basis for the Morse decomposition part of the Conley theory for control systems (as references sources we mention the papers [22, 23, 25, 27, 83, 94, 95, 96, 97, 99, 100, 101, 102]).

The present book unites the ideas from these papers and the Colonius–Kliemann book to define the elements of the Conley theory for control systems. Although dynamical concepts are easily generalized by the semigroup methodology, control systems need not satisfy important conditions as, for instance, invariance of limit sets and attractors. Thus, extending Conley results from dynamical systems to control systems is a nontrivial work. Faced with the possibility of noninvariant attractors and repellers, two distinct notions of Morse decomposition have been considered in the control system set up ([25, 96]). A 'dynamic Morse decomposition' is a finite collection of compact isolated invariant sets (Morse sets) which consist of the residence of limit sets, while the trajectories of an external point can not come near the same Morse set for both forward and backward times. An 'attractor-repeller Morse decomposition' is a finite collection of sets given by intersections of sets in a sequence of attractors and complementary repellers. In a classical dynamical system, Conley shows that these two notions of Morse decomposition are equivalent. In the paper [25], the Conley results were proved in a special situation where control systems satisfied certain translation hypothesis. One of the main tasks in this book is to relate dynamic and attractor-repeller Morse decompositions for control systems without assuming the translation hypothesis. The strategy is based on invariance issues of attractor-repeller pairs. In compact state space, one verifies that any dynamic Morse decomposition has an attractor-repeller configuration. On the other hand, Morse decompositions determined by invariant attractor-repeller pairs are dynamic Morse decompositions. Thus, dynamic and attractor-repeller Morse decompositions are equivalent concepts for control systems with invariant attractor-repeller pairs.

The other principal intention of this book is to reproduce the Conley theorem that connects Morse decomposition and chain recurrence. Every attractor-repeller Morse decomposition contains the chain recurrence set, which is the set of all chain recurrent points of the control system. In a special case, if the finest attractor-repeller Morse decomposition exists, then its Morse sets coincide with the connected components of the chain recurrence set (or, simply, chain components). On the other hand, if the chain recurrence set admits finitely many connected components then it determines a dynamic Morse decomposition. Consequently, for a control system with invariant attractor-repeller pairs, the existence of finitely many chain components is a necessary and sufficient condition for the existence of the finest Morse decomposition. The Conley theorems are concluded by constructing a Lyapunov function for Morse decomposition and then proving the existence of a complete Lyapunov function for control systems on compact manifolds.

Many things should be explained before proving the Conley theorems. This effort provides particular results which are interesting in themselves. In fact, outside a Morse decomposition, every trajectory of the system comes close to some Morse set, in forward as well as in backward time. Thus the control system does not admit asymptotic transitivity outside a Morse decomposition, or in other words, any asymptotic transitive set resides in some Morse set. This means that periodicity, recurrence, and controllability occur in the components of a Morse decomposition. In view of this fact, full chapters are dedicated to study all transitivity concepts, with the main intention being to describe technically what happens inside Morse sets. The amount of information about Morse decomposition will be completed with the study of chain transitivity, contributing to an estimate of the global dynamic behavior of a control system.

In view of the attractor-repeller configuration of Morse decomposition, a full chapter is devoted to explaining attractors. There are various notions of attraction in the theory of dynamical systems. The notion of an attractor for a singular point was used by E. Coddington and N. Levinson [32]. The conception of an attractor for a closed set was first studied by J. Auslander, N. Bhatia, and P. Siebert [4]. The concepts of attractors related to stability theory were extensively studied by N. Bhatia *et al.* [11, 12, 13, 14], including the notions of global weak attractors and global uniform attractors. Alternative concepts of attractors and global attractors were used by J. Hale [56] and O. Ladyzhenskaya [70]. Finally, C. Conley [36] defined a special notion of an attractor that generates Morse decompositions. All these concepts of attractors are studied in the control system framework. The main task consists of proving the connection between the Conley attractor and the uniform attractor. In general, every compact Conley attractor is an asymptotically stable set, which means a stable uniform attractor. This result yields an important statement that the existence of the finest Morse decomposition implies the existence of a chain component that is asymptotically stable.

This book has been written with a wide audience in mind: control-theoretic engineers or mathematicians, post-graduate students, and graduate students researching dynamical systems and geometry. Control theorists may go directly to Chapter 4. Readers who are not familiar with dynamical systems are invited to consult Appendix A. The two first chapters of the book contain elementary concepts of control affine systems, but they are not mere preliminaries. Chapter 1 provides an introduction to the basic definitions and properties of control affine systems. It presents detailed mathematical formulations of integral curves, shift space, control flow, and system semigroup. Chapter 2 studies the elementary concepts of invariance, orbits, equilibria, and periodicity in the control system setting. Topological properties of invariant sets and orbits are investigated. Characterizations of critical and periodic points are presented with some new features (Theorems 2.3.1, 2.3.2, 1.2.1, and Proposition 2.4.1).

The middle part of this book consists of studies into various aspects of transitivity for control systems. The weak transitivity relation is defined together with the concept of minimal sets in Chapter 3. The main result shows that the minimal sets are upper bounds for a dynamic order among the equivalence classes of weak transitivity. Chapter 4 treats the concepts of limit sets and prolongational limit sets for control systems. These are crucial for the concepts of asymptotic transitivity and attraction. The notion of prolongation is used to describe compact equistable sets. Special attention is given to minimal equistable sets, an extension of minimal sets. The notion of asymptotic transitivity is presented in Chapter 5 as a relation among the Poincaré recurrent points of the control system. The studies include the classical control sets and invariant control sets. A state point is positively recurrent if it lies in its positive limit set. Two positively recurrent points are equivalent if each one is a limit point of the other. This relation constitutes the asymptotic transitivity and is proved to be an equivalence relation among the positively recurrent points. The Poincaré recurrence theorem for control systems is reproduced, stating that μ-almost every point is positively recurrent, with μ an invariant probability measure. The notion of a nonwandering point is also studied in Chapter 5. By definition, a nonwandering point lies in its prolongational limit set. The main result shows that every point closed to a recurrent point is nonwandering. In view of this relation, a more general notion of controllability is defined by prolongations and its relative prolongational control sets are studied.

A new feature in this middle section concerns the relation between Poincaré recurrence and periodicity. It is clear that a periodic point is recurrent. The converse does not hold, except in very special situations. The Poincaré–Bendixson theorem states that a nonempty compact limit set of a C^1 planar dynamical system, which contains no equilibrium point, is a periodic trajectory ([60, Chapter 11]). In a higher dimension, however, it has no generalization or counterpart. A famous example of dynamical system on the 2-torus shows recurrent points which are not periodic (see a version for control system in Example 5.2.1). In the general set-up of control systems on manifolds, recurrence and periodicity are equivalent concepts for points with closed semi-orbits (Theorems 5.2.1, 5.2.2, 5.2.3, and 5.2.4).

Attractors, Morse decompositions, and chain transitive sets make up the main part of the book. Chapter 6 deals with various notions of attractors and repellers for control systems. Since the Conley definition of attractor approaches the uniform attraction, one gives special attention to the properties of uniform attractor. Both the notions of global attractor and global uniform attractor are studied in the chapter. A measure of noncompactness is used to describe the asymptotic behavior of a control system admitting a global attractor. Conditions for the existence of a global attractor are discussed. Conley theorems on chain recurrence are presented in Chapter 7. By contemplating trajectories with jumps, the chain transitivity generalizes the asymptotic transitivity. This is also an equivalence relation and its equivalence classes are described as maximal regions of chain transitivity. One shows that asymptotic transitive and prolongational control sets are contained in some chain transitive set. In order to present the main Conley theorem of chain recurrence, the attractor-repeller pair paradigm is defined. The Conley theorem describes the chain recurrence set as the intersection of all attractor-repeller pairs of the control system. The last chapter of the book studies dynamic and attractor-repeller Morse decompositions of control systems separately. In compact state space, every dynamic Morse decomposition admits an attractor-repeller configuration, while, on the other hand, every invariant attractor-repeller Morse decomposition is a dynamic Morse decomposition. The main Conley theorem of chain recurrence implies that any Morse decomposition contains all chain recurrent points of the system. The existence of finitely many chain transitivity classes is equivalent to the existence of the finest Morse decomposition. The last important result shows that the components of a Morse decomposition are connected by trajectories which go through decreasing levels of some Lyapunov function. This implies the existence of a complete Lyapunov function that decreases strictly on trajectories outside the chain recurrence set and maps each chain equivalence class onto a critical value.

Although these theorems on Morse decompositions are not new, they are proved with the absence of translation hypothesis. This extends and improves the results on dynamic Morse theory for control systems. The notion of a limit compact control system is a new, and the relation between asymptotically compact and limit compact control systems is thus a new result (Proposition 6.4.6). Another new feature is the use of a prolongational limit set to describe the complementary repeller of a Conley attractor as well as the properties of attractor-repeller pair (Theorem 7.2.3 and Proposition 7.2.4). Finally, it was not known that nonwandering points are chain recurrent points (Proposition 7.1.7).

The dynamical concepts of a control system do not require compact state space, but the main Conley theorems are assured under compactness. The dynamic Morse theory on noncompact space the has many open questions to be investigated. At the end of Chapter 8, a formulation of generalized Morse decomposition is suggested for further discussion on Conley's ideas in noncompact state space.

Parts of the book may be used for one-semester courses or seminars in mathematics or control-theoretic engineering, with the following suggestions:

- A first course in control system theory: Chapters 1 and 2, for students who are familiar with ordinary differential equations and basic notions of real analysis and general topology.

- Controllability and asymptotic transitivity: Chapters 3, 4, and 5.
- Morse decompositions and chain transitivity: Chapters 6, 7, and 8.
- Seminar on periodicity: Sections 2.2, 2.3, 2.4, and 5.1.
- Seminar on Poincaré recurrence: Sections 4.1, 5.2, and 5.3.
- Seminar on attractors: Chapter 6.
- Introduction to Morse theory of dynamical systems: Appendix A.

The book may be also used as a reference source for various topics in dynamical and control systems, contributing to the research of students interested in the dynamics of control systems. Its contents may integrate basic references for academic dissertations.

Implicitly, the book contains a survey about ongoing research into the dynamics of control systems. It reproduces the Morse decomposition part of the Conley theory, leaving the Conley index theorems to future work. The dynamical concepts presented here might be viewed from different mathematical problems. For instance, the family of differential equations given by the formula $\dot{x} = X_0(x) + \sum_{i=1}^{n} u_i(t) X_i(x)$ can be interpreted as time-dependent perturbations of an ordinary differential equation by the vector fields $X_1, ..., X_n$. In this case, one wants to study the Conley concepts of the perturbed system relative to the Conley concepts of the nominal dynamical system $\dot{x} = X_0(x)$. An advanced mathematical problem considers $X_0, ..., X_n$ as invariant vector fields on a Lie group. A task in this case is to investigate a homogeneous structure for Morse decompositions and chain transitivity. Another mathematical problem discusses the relationship between periodicity and Poincaré recurrence of control systems with the intention of establishing a general Poincaré-Bendixson theorem.

Chapter 1

Fundamental Theory of Control systems

In the language of mathematics, the concept of a control system was formulated with the purpose of regulating dynamical systems. Intuitively, a control system is an undetermined dynamical system, in which appropriate functions can be chosen to a given criterion, resulting in a system with desired properties. The possible model classes of control systems are, for instance, algebraic differential models (e.g., Flies *et al.* [49]), input-output operators (e.g., Francis [51]), the behavioral approach (e.g., Willems [113]), or the classical system model (e.g., Colonius and Kliemann [34]). The classical mathematical formulation of a control system consists of a family of differential equations, which are interpreted as models for distinct forms of operation of the same device. A set of control functions (inputs) determines the differential equations and then the behaviors (outputs) of the system.

This book considers the classical mathematical paradigm of control systems. This first chapter introduces the basic definitions and presents the main properties of control systems used afterwards. Section 1.1 contains the general definition of control systems. Section 1.2 defines the shift space, which often determines the control index set. Section 1.3 introduces the class of control affine systems, which is the model of control system studied throughout the book. The flow point of view of a control affine system is described in Section 1.4 via the control flow. Section 1.5 describes the group structure of a control affine system.

1.1 Basic definitions

This first section contains basic notations and definitions of control systems. Throughout the book, the symbol M denotes a differentiable manifold equipped with a compatible distance d; $\mathbb{R}^n$ stands for the n-dimensional Euclidean space endowed with an inner product $\langle\cdot,\cdot\rangle$ and associated norm $\|\cdot\|$.

The state space of a control system consists of a manifold M, where we often deal with topological objects. In order to establish the basic notations for topological concepts, for $x \in M$ and $\epsilon > 0$, we define the open ϵ-ball $\mathrm{B}(x,\epsilon)$ and the closed ϵ-ball $\mathrm{B}[x,\epsilon]$ centered at x respectively as

$$\begin{aligned}\mathrm{B}(x,\epsilon) &= \{y \in M : \mathrm{d}(x,y) < \epsilon\},\\ \mathrm{B}[x,\epsilon] &= \{y \in M : \mathrm{d}(x,y) \leq \epsilon\}.\end{aligned}$$

For a given subset $X \subset M$, the notations $\mathrm{int}(X)$, $\mathrm{cl}(X)$, and $\mathrm{fr}(X)$ means respectively the interior of X, the closure of X, and the boundary of X in M. The same notations are used in any other case of metric space that appears throughout the book.

In order to define admissible control inputs, we need the following ingredients.

Definition 1.1.1 *For a given real number $s \in \mathbb{R}$ and two functions $u, v : \mathbb{R} \to \mathbb{R}^n$, the s-**concatenation** of u and v is the function $w : \mathbb{R} \to \mathbb{R}^n$ defined by*

$$w(t) = \begin{cases} u(t), & \text{if } t \leqslant s \\ v(t-s), & \text{if } t > s \end{cases}.$$

*The s-**shift** of u is the function $u \cdot s : \mathbb{R} \to \mathbb{R}^n$ given by $u \cdot s\,(t) = u\,(s+t)$ for all $t \in \mathbb{R}$.*

For sequences of functions $u_1, ..., u_k$ and numbers $s_1, s_2, ..., s_{k-1}$ with $s_1 < s_2 < ... < s_{k-1}$, we may define the $(s_1, ..., s_{k-1})$-concatenation of $u_1, ..., u_k$ by

$$u(t) = \begin{cases} u_1(t), & \text{if } t \leqslant s_1 \\ u_2(t-s_1), & \text{if } s_1 < t \leq s_2 \\ \vdots & \\ u_k(t-s_{k-1}), & \text{if } t > s_{k-1} \end{cases}.$$

Definition 1.1.2 *Let $\mathrm{U} \subset \mathbb{R}^n$. A family of functions $\mathcal{U} = \{u : \mathbb{R} \to \mathrm{U}\}$ is said to be **admissible** if it satisfies the following properties:*

1. *Each function $u \in \mathcal{U}$ is locally integrable, that is, u is Lebesgue integrable on every bounded interval.*

2. *For each $u \in \mathcal{U}$ and $s \in \mathbb{R}$, the s-shift $u \cdot s$ is contained in $\mathcal{U}$.*

3. *For $u, v \in \mathcal{U}$ and $s \in \mathbb{R}$, the s-concatenation of u and v is contained in $\mathcal{U}$.*

A trivial admissible family is given by one constant function $u(t) \equiv u_0 \in \mathbb{R}^n$. A non-trivial admissible family is basically formed by piecewise constant functions. A function $u : \mathbb{R} \to \mathrm{U}$ to be piecewise constant means that the real line $\mathbb{R}$ is decomposed into subintervals of length bounded below by a positive number such that u is constant on each subinterval. The admissible family of all piecewise constant functions is denoted by $\mathcal{U}_{pc}$.

The general definition of control systems is given in the following.

Definition 1.1.3 *A **control system** $\Sigma = (M, \mathrm{U}, \mathcal{U}, \mathrm{X})$ is formed by*

1. *A state space M that is a d-dimensional C^∞-differentiable manifold.*

2. *A control range $\mathrm{U} \subset \mathbb{R}^n$ and an admissible family of control functions $\mathcal{U} = \{u : \mathbb{R} \to \mathrm{U}\}$.*

3. *A family of ordinary differential equations*

$$\dot{x} = \mathrm{X}(x, u(t))$$

depending on the control functions, where $\mathrm{X} : M \times \mathbb{R}^n \to TM$ is a C^∞ map from the product manifold $M \times \mathbb{R}^n$ into the tangent bundle TM of M.

In the case $M = \mathbb{R}^d$, one may consider a C^∞ map $\mathrm{X} : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^d$ in the definition of control systems.

Then each element u in the control range U corresponds to a constant control function. In the trivial case of a unique constant control function, the control system corresponds to a classical dynamical system. In the nontrivial case, each control function $u \in \mathcal{U}$ determines a nonautonomous differential equation $\dot{x} = \mathrm{X}(x, u(t))$ on the manifold M. To guarantee solutions for this equation, we can not apply the usual procedure to reduce the nonautonomous differential equation to an autonomous one by introducing t as an additional state variable, since the dependence of u on t need not be differentiable. However, we may locally analyze the associated integrated equation

$$x(t) = x + \int_0^t \mathrm{X}(x(s), u(s))\ ds$$

and proceed with Carathéodory's theory allowing measurable dependence on t. In this case, a solution $x(t)$ of the differential equation $\dot{x} = \mathrm{X}(x, u(t))$ with initial condition $x(0) = x_0$ is a broken integral curve, that is, $x(t)$ is absolutely continuous, satisfies the differential equation almost everywhere, and satisfies the initial condition. Here, $x(t)$ absolutely continuous means that $x(t)$ is differentiable almost everywhere, its derivative is Lebesgue integrable, and $x(t) = x(0) + \int_0^t x'(s)\ ds$. We refer to Sontag [92] for the technical details in $\mathbb{R}^d$ and to Coddington [32] for Carathéodory's theory of differential equations.

The theory developed in this book requires solutions defined forward on all positive times $t \in \mathbb{R}^+$. In fact, we study concepts which depend on the action of a semigroup defined by positive time transitions. This is explained in Section 1.5 and commented in the notes throughout the book. For technical simplifications, however, we assume throughout that for each control function $u \in \mathcal{U}$ and each point $x \in M$ the preceding equation has a unique solution $\varphi(t, x, u)$, defined for all time $t \in \mathbb{R}$, with $\varphi(0, x, u) = x$. In particular, we assume that for every u in the control range U the C^∞ vector field $\mathrm{X}_u : x \in M \mapsto \mathrm{X}(x, u) \in TM$ is complete, that is, the corresponding flow $(t, x) \to \varphi(t, x, u)$ is defined globally on $\mathbb{R} \times M$.

Three special classes of control systems are extensively studied in mathematics. They are:

Example 1.1.1 *A **linear control system** takes the form*

$$\dot{x} = A(x) + B(u(t))$$

where A is a $d \times d$ real matrix, B is a $d \times n$ real matrix, $M = \mathbb{R}^d$, and $\mathrm{U} = \mathbb{R}^n$.

Example 1.1.2 *A **bilinear control system** takes the form*

$$\dot{x} = A_0(x) + \sum_{i=1}^{n} u_i(t) A_i(x)$$

where $u(t) = (u_1(t), ..., u_n(t)) \in \mathbb{R}^n$, $A_0, A_1, ..., A_n$ are $d \times d$ real matrices, $M = \mathbb{R}^d$, and $\mathrm{U} = \mathbb{R}^n$.

Example 1.1.3 *A **control affine system** on a C^{∞}-manifold M takes the form*

$$\dot{x} = X_0(x) + \sum_{i=1}^{n} u_i(t) X_i(x)$$

where $X_0, X_1, ..., X_n$ are C^{∞}-vector fields on M. Both linear and bilinear control systems are special cases of control affine system.

1.2 Shift Space

We shall now describe the shift space, the theoretic set of control inputs considered throughout the book. It is a metrizable compactification of the piecewise control functions and the shift map defines a dynamical system on it. The results presented in this section are extracted from [34, Chapter 4]. Readers who are not familiar with concepts of functional analysis may consult [84] or just consider the statements as preliminary assumptions.

Let $\mathcal{U}_{pc}$ be a family of piecewise constant functions with a compact and convex control range $\mathrm{U} \subset \mathbb{R}^n$. Let $L_\infty(\mathbb{R}, \mathbb{R}^n)$ be the vector space of all measurable functions from $\mathbb{R}$ into $\mathbb{R}^n$ which are essentially bounded, i.e. bounded up to a set of measure zero. Since the control range $\mathrm{U} \subset \mathbb{R}^n$ is compact, we have $\mathcal{U}_{pc} \subset L_\infty(\mathbb{R}, \mathbb{R}^n)$. It is well-known that $L_\infty(\mathbb{R}, \mathbb{R}^n) = L_1(\mathbb{R}, \mathbb{R}^n)^*$, where $L_1(\mathbb{R}, \mathbb{R}^n)$ is the space of all functions from $\mathbb{R}$ into $\mathbb{R}^n$ for which the absolute value is Lebesgue integrable, and $L_1(\mathbb{R}, \mathbb{R}^n)^*$ is the dual space of $L_1(\mathbb{R}, \mathbb{R}^n)$ (see e.g. [33, Theorem 4.5.1]). For each $\alpha \in L_1(\mathbb{R}, \mathbb{R}^n)$, the L_1-norm of α is given by

$$\|\alpha\|_1 = \int_{\mathbb{R}} \|\alpha(t)\| \ dt.$$

Define the linear functional $f_\alpha : L_\infty(\mathbb{R}, \mathbb{R}^n) \to \mathbb{R}$ by

$$f_\alpha(u) = \int_{\mathbb{R}} \langle u(t), \alpha(t) \rangle \ dt.$$

The weak* topology on $L_\infty(\mathbb{R}, \mathbb{R}^n)$ is the weak topology induced on $L_\infty(\mathbb{R}, \mathbb{R}^n)$ by the collection $\{f_\alpha : \alpha \in L_1(\mathbb{R}, \mathbb{R}^n)\}$, that is, the weakest topology such that f_α is continuous for all $\alpha \in L_1(\mathbb{R}, \mathbb{R}^n)$. This is the topology for which the sets $f_\alpha^{-1}((a, b))$, for $\alpha \in L_1(\mathbb{R}, \mathbb{R}^n)$ and (a, b) open interval in $\mathbb{R}$, form a subbase.

From now on, we assume that $L_\infty(\mathbb{R}, \mathbb{R}^n)$ is endowed with the weak* topology and define the subspace $\mathcal{U} = \mathrm{cl}(\mathcal{U}_{pc}) \subset L_\infty(\mathbb{R}, \mathbb{R}^n)$.

Lemma 1.2.1 *The shift is an internal operation in* $\mathcal{U}$.

Proof. Let $u \in \mathcal{U}$ and $t \in \mathbb{R}$. Suppose that $u \cdot t \in \bigcap_{i=1}^{k} f_{\alpha_i}^{-1}((a_i, b_i))$ with $\alpha_1, ..., \alpha_k \in L_1(\mathbb{R}, \mathbb{R}^n)$. Then $f_{\alpha_i}(u \cdot t) \in (a_i, b_i)$ for each i. Since

$$f_{\alpha_i}(u \cdot t) = \int_{\mathbb{R}} \langle u(t+s), \alpha_i(s) \rangle \, ds = \int_{\mathbb{R}} \langle u(\tau), \alpha_i(\tau - t) \rangle \, d\tau = f_{\alpha_i \cdot (-t)}(u)$$

we have $u \in \bigcap_{i=1}^{k} f_{\alpha_i \cdot (-t)}^{-1}((a_i, b_i))$. Hence there is some $v \in \mathcal{U}_{pc}$ such that $v \in \bigcap_{i=1}^{k} f_{\alpha_i \cdot (-t)}^{-1}((a_i, b_i))$. This implies that $v \cdot t \in \bigcap_{i=1}^{k} f_{\alpha_i}^{-1}((a_i, b_i))$. Since $v \cdot t \in \mathcal{U}_{pc}$, it follows that $u \cdot t \in \mathcal{U}$. □

This enables us to define the shift map $\Theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U}$ by $\Theta(t, u) = u \cdot t$. The **shift space** is given by the space $\mathcal{U}$ equipped with the shift map. We can show that the shift space is a compact metrizable space and the shift map is continuous. We need the following:

Lemma 1.2.2 *The subspace* $\mathfrak{U} \subset L_\infty(\mathbb{R}, \mathbb{R}^n)$ *given by*

$$\mathfrak{U} = \{u : \mathbb{R} \to \mathbb{R}^n : u(t) \in \mathrm{U} \text{ for a.a. } t \in \mathbb{R}, \text{ measurable}\}$$

is a compact metrizable space.

Proof. It should be noticed that $L_1(\mathbb{R}, \mathbb{R}^n)$ is a separable Banach space, which means there is a countable and dense subset $\{\alpha_k : k \in \mathbb{N}\}$ of $L_1(\mathbb{R}, \mathbb{R}^n)$ (see [33, Proposition 3.4.5]). Define

$$d(u, v) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|f_{\alpha_k}(u - v)|}{1 + |f_{\alpha_k}(u - v)|} \tag{1.1}$$

for every pair $u, v \in L_\infty(\mathbb{R}, \mathbb{R}^n)$. By Alaoglu's theorem, the unit ball of $L_\infty(\mathbb{R}, \mathbb{R}^n)$ is compact and metrizable, and a metric is given by 1.1 (see [44, Theorem 3, p. 434]). Since the control range U is compact, the set $\mathfrak{U}$ is bounded in $L_\infty(\mathbb{R}, \mathbb{R}^n)$. It remains to show that $\mathfrak{U}$ is closed (weak* closed) in $L_\infty(\mathbb{R}, \mathbb{R}^n)$. Indeed, for any compact interval $I \subset \mathbb{R}$, consider the set

$$\mathfrak{U}|_I = \{u|_I : u \in \mathfrak{U}\} \subset L_2(I, \mathbb{R}^n)$$

where $L_2(I, \mathbb{R}^n)$ is endowed with the L_2-norm given by $\|u\|_2 = \int_{\mathbb{R}} \|u(t)\|^2 \, dt$. Since U is convex, $\mathfrak{U}|_I$ is a convex set in $L_2(I, \mathbb{R}^n)$. Moreover, for a given sequence (u_n) in $\mathfrak{U}|_I$ that converges to $u \in L_2(I, \mathbb{R}^n)$ in the L_2-norm, there