The Analysis of Selected Algorithms for the Stochastic Paradigm

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By

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I am deeply and forever indebted to my parents Miled and Rebecca, my brother Maroun and my sister Zeina, and to all my family for their love, support, and encouragement throughout my entire life and without whom this work would not have been accomplished…

TABLE OF CONTENTS

PREFACE

This book is entitled *The Analysis of Selected Algorithms for the Stochastic Paradigm* and it includes the analysis and selected algorithms for random and stochastic phenomena in the following areas: basic probability, random variables, mathematical expectation, special probability and statistical distributions, random processes, Markov chains, in addition to the presentation of my "Complex Probability Paradigm" applied to the Brownian motion. Why algorithms and why probability? My background is a *Ph.D.* in Computer Science and a *Ph.D.* in Applied Mathematics, so I combined my knowledge in computers and Applied Mathematics with Probability theory and I wrote this manuscript. Eager to learn and discover more as well as to apply my knowledge in mathematics and computer science, I embarked on this new research on stochastic phenomena.

Additionally, each time I work on this field I find the pleasure to tackle the knowledge, the theorems, the proofs, and the applications of the theory of probability. In fact, each problem on probability is like a riddle to be solved, a conquest to be won, and I become relieved and extremely happy when I reach the end of the solution. This verily proves two important facts: firstly, the power of mathematics and its models to deal with such kind of problems and secondly the power of the human mind that is able to understand such class of problems and to tame such a wild concept that is randomness, probability, stochasticity, uncertainty, chaos, and chance.

Mathematical probability is an attractive, thriving, and respectable part of mathematics. Some mathematicians and philosophers of science say the gateway to mathematics deepest mysteries. Moreover, mathematical statistics denotes an accumulation of mathematical discussions connected with the efforts to most efficiently collect and use numerical data subject to random variation. In the twentieth century and present time, the concept of probability and mathematical statistics has become one of the fundamental notions of modern science and philosophy of nature. This was accomplished after a long history of efforts done by prominent and distinguished mathematicians and philosophers like the famous French *Blaise Pascal* and *Pierre de Fermat*, the Dutch *Christiaan Huyghens*, the Swiss *Jakob Bernoulli*, the German *Carl Friedrich Gauss*, the French *Siméon-Denis*

Poisson, the English *Thomas Bayes*, the French *Joseph Louis Lagrange* and *Pierre-Simon Laplace*, the English *Karl Pearson* and *Ronald Aylmer Fisher*, the Russian *Andrey Nikolaevich Kolmogorov*, the American *John von Neumann*, etc…

As a matter of fact, each time I read or meditate these outstanding giants I feel the respect, the admiration, and the esteem to these magnificent men and giants of science who most of them were mathematicians, physicists, astronomers, statisticians, philosophers, etc... at the same time. They were, as we call them today: *Universalists*.

The fields to which this book belongs to are that of Probability and Computer Science, hence the present work should – and it certainly does – include applications to both fields that encompass a wide set of problems taken from engineering, games of chance (cards, dice, urns, coins, etc…), fundamental mathematics, computer science, physics…

To delve into the mysteries of the stochastic paradigm, ten chapters and sixty-five algorithms were written for this purpose, and are devoted to the illustration of probability, random variables, and stochastic processes concepts and theory.

What is original in this work, like the title of the book suggests, is firstly the analysis of a large array of stochastic problems and secondly the sixty-five algorithms written for this array of applications. Both are my creation and they illustrate how probability theory can be applied to solve and understand random phenomena existent in innumerable branches of science and pertaining to different disciplines of knowledge. Though, this was not certainly possible without the help of all the books, websites, and encyclopedias that provided me with a very rich source of hints, information, and mathematics. Hence, in some parts of the manuscript I have used the wording of the initial theorems, or the problems given, or sometimes the methods of proof, always avoiding committing plagiarism and with the intention to preserve the integrity and the truth of the data taken from the reference books mentioned in the bibliography. However, the algorithms are completely the result of a personal effort.

Moreover, the book develops methods for simulating simple or complicated processes or phenomena. If the computer can be made to imitate an experiment or a process, then by repeating the computer simulation with different data, we can draw statistical conclusions. Thus, a

simulation of a wide spectrum of random processes on computers was done. The result and accuracy of all the algorithms are truly amazing and delightful; hence, this confirms two complementary accomplishments: first the triumphs of the theoretical calculations already established using different theorems and second the power and success of modern computers to verify them.

The work was done using *Microsoft Visual C++* due to its excellent well-definedness, modularity, portability, and efficiency. Moreover, they were executed on a workstation computer with parallel microprocessors to acquire a suitable speed and efficiency needed for these numerical and computational methods.

Chapter IX is an additional section that was added to this book and is entitled: The Complex Probability Paradigm and the Brownian Motion. It is the development of *A. N. Kolomogorov*'s system of axioms and is applied to the Brownian motion. The theorem is my creation also and my new mathematical paradigm that I called "*The Complex Probability Paradigm*" was the subject of a personal and published twelve research papers since 2010 to 2018. I wrote five algorithms to illustrate it.

To conclude, due to its universality, mathematics is the most positive and certain branch of science. It is successfully called by philosophers the *Esperanto of all sciences* since it is the common, the logical, and the exact language of understanding, capable of expressing accurately all scientific endeavor. Although Probability and Statistics are approximate sciences that deal with rough guesses, hypotheses tests, estimated computations, expected calculations, and uncertain results, they still keep in them the spirit of "exact" sciences through their numbers, proofs, figures, and graphs, since they remain to be a branch of mathematics. Surely, the pleasure of working and doing mathematics and computer science is everlasting. I hope that the reader will benefit from both and share the pleasure of examining the present manuscript. As a matter of fact, the combination of both mathematics and computer science leads to "magical" and amazing results and algorithms, and the following work is an illustration of this approach.

Sincerely, I am truly astonished by the power of statistics and probability to deal with random data and phenomena, and this feeling and impression never left me from the first time I was introduced to this branch of science and mathematics. I hope that in the present book I will convey and share

this feeling with the reader. I hope also that he will discover and learn about the concepts and applications of its paradigm.

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Notre Dame University-Louaizé, Lebanon
August 17th, 2019.

CHAPTER I

INTRODUCTION

> "*The most incomprehensible thing about the universe is that it is comprehensible…"*
>
> *Albert Einstein.*

> *"One thing I have learned in a long life*: *That all our science, measured against reality, is primitive and childlike – and yet it is the most precious thing we have."*
>
> *Albert Einstein.*

It gives me great pleasure to introduce as well as to discuss, to learn, to solve, to teach, and to work with probability and stochastic theories. So, let us first start the introduction of the present book with a relatively small historical note that will give us an overview on the development of this fascinating paradigm across the ages till modern times, before thoroughly studying its essential theorems and examining its diverse applications.

Probability theory is a branch of statistics, a science that employs mathematical methods of collection, organization, and interpretation of data, with applications in practically all scientific areas. When working with probability theory, we analyze random – or stochastic – phenomena and assess the likelihood that an event will occur. In this book we discuss some fundamental aspects of probability theory and explore their use to solve a large array of problems as we will see later.

One of the earliest mathematical studies on probability was *Liber de ludo aleae* ("On Casting the Die"), written by the 16th–century Italian mathematician and physicist **Gerolamo Cardano** (1501-1576); it was not published until 1663, 87 years after Cardano's death. Cardano introduced the concepts of combinatorics into the calculations of probability and defined probability as "the number of favorable outcomes divided by the number of possible outcomes". It is likely that Cardano would be known as "the father of the theory of probability" had the publication not been delayed.

In the 17th century, questions about the probability of events occurring in games of chance were discussed in the correspondence between the French mathematicians **Blaise Pascal** (1623-1662) and **Pierre de Fermat** (1601-1665). Building on their results, the Dutch physicist-astronomer-mathematician **Christiaan Huyghens** (1629-1695) published, in 1656, *De ratiociniis in ludo aleae* ("On Reasoning in Games of Chance"). Moreover, the Swiss mathematician **Jakob Bernoulli** (1654-1705) was an early advocate of the use of probability theory in medicine and meteorology in his work *Ars conjectandi* ("The Art of Conjecture"), published posthumously in 1713.

In the late 18th century, it became increasingly evident that analogies exist between games of chance and random phenomena in physical, biological, and social sciences. Contributions of fundamental importance to probability theory were made in the latter half of the 18th century and the beginning of the 19th century by the French mathematicians, astronomers, and physicists **Joseph Louis Lagrange** (1736-1813) and **Pierre-Simon Laplace** (1749-1827), the omnipresent mathematician and astronomer **Carl Friedrich Gauss** (1777-1855), and the French mathematician **Siméon-Denis Poisson** (1781-1840). The most important publication on probability theory in this era is Laplace's *Théorie analytique des probabilités* (1812), which discussed practical applications of the theory and developed the concepts of a normal distribution, first discovered by **Abraham de Moivre** (1667-1754). In the 1837 *Recherches sur la probabilité des jugements…* ("Researches on the Probability of Opinions…") Poisson introduced what we now know as the Poisson distribution, or Poisson law of large numbers, an approximate method used to describe probable occurrence of unlikely events in a large number of unconnected trials.

About 1850-1900, the Russian (Petersburg) school of probability theory, emphasizing stringent mathematical methods, dominated its development. Prominent figures of this school were **Pafnuty Chebyshev** (1821-1894) and, originally disciples of Chebyshev's, **Andrei Markov** (1856-1922) and **Alexandr Lyapunov** (1857-1918).

In the beginning of the 20th century, the need for applications of probability theory increased in physics, economics, insurance, and telephone communication. Important impulses were given by **Albert Einstein** (1879-1955), the New Zealand-born English physicist **Ernest Rutherford** (1871-1937), and the Swedish astronomer **Carl Vilhelm Ludwig Charlier** (1864-1934). Applications often precipitated new probability problems which had

to be tackled within the field of theoretical probability, and thus a fruitful interplay between the sciences was created.

Moreover, the English mathematician **Karl Pearson** (1857-1936) is the founder of modern hypothesis testing – he developed the chi-squared test of statistical significance. Besides making major contributions in mathematics and probability theory, Pearson also practiced law, was active in politics, published literary works, and wrote *The Grammar of Science* (1892), a classic in the philosophy of science.

In addition, one of the most eminent scientists of the 20th century, the English geneticist and statistician **Ronald Aylmer Fisher** (1890-1962), professor of genetics at Cambridge from 1943 to 1957, developed methods of *multivariate analysis* – analysis of problems involving more than one variable – and used them in his investigations of the linkage of genes to various traits. He also introduced the idea of likelihood in statistical inference, that is, how to draw conclusions on the basis of the relative probability of events. Fisher's *Statistical Methods for Research Works* (1925) was used extensively as a textbook and a reference book and remained in print for more than 50 years. Along with **Egon Pearson** (1895-1980) – the son of Karl Pearson – and Fisher, **Jerzy Neyman** (1894-1981) was one of the principle founders of modern statistical analysis. Neyman lived in Poland until he was forty, then in England, and finally settled in the United States. In 1955, he became professor of statistics at the University of California at Berkeley, where his department became a world center for the development of mathematical statistics. Egon Pearson and Neyman founded what is now known as the school of statistical inference.

Furthermore, the Russian mathematicians **Alexandr Khinchin** (1894-1959) and **Andrey Nikolaevich Kolmogorov** (1903-1987) are the founders of the Moscow school of probability theory, one of the most influential in the 20th century. One may say that the present golden age of probability theory started in 1933 with Kolmogorov's *Grundbegriffe der Wahrscheinlichkeitsrechnung* ("Foundations of Probability Theory"). Kolmogorov introduced several fundamental postulates in statistics and probability theory; he showed that probability theory may be founded on the concepts of set theory and mathematical measure theory.

Also, the theory of games was founded by **John von Neumann** (1903-1957), pre-eminent 20th century innovator in many fields of pure and applied mathematics. He created a mathematical model for games of chance, such

as poker and bridge, that involve free choices – strategy – for the players. His first paper on this subject was presented in 1926; von Neumann's theories were further developed in his major work *Theory of Games and Economic Behavior* (1944), co-authored with the economist **Oskar Morgenstern** (1902-1977). The theory of games is now a mathematical discipline of its own, with far-reaching applications to economics and social sciences.

Additionally, I chose the word paradigm for this branch of mathematical sciences after consulting the influential book of the historian of science **Thomas Kuhn**, which is *The Structure of Scientific Revolutions*, where the author used the term to describe a set of theories, standards, and methods that together represent a way of organizing knowledge – that is, a model or a way of viewing the world. Kuhn stated in his thesis that revolutions in science occur when an older paradigm is reexamined, rejected, and replaced by another, just like Einstein's theories of special and general relativity that dethroned Newtonian mechanistic theory, or quantum mechanics that replaced the classical theories of electromagnetism and thermodynamics when probing the micro-world…What about probability and statistics? We can affirm that their set of theories and methods developed across the centuries have defined for us a way to view the world, techniques and a model to understand and to deal with such concepts as randomness, chance, stochasticity, chaos, probability…Hence, to be brief, the definition of a paradigm suits very well this discipline of knowledge and this methodology of thinking. This justifies my usage of this term.

After this historical introduction and the last note that followed it, we show now briefly the structure of the book which is divided into 10 chapters. Hence, the structure is as follows:

Chapter I, is an introduction to the book that starts with a historical note then states the basic ideas and algorithms that will be developed throughout the whole manuscript.

Chapter II, is an introductory chapter to define the fundamental mathematical concepts and methods that will be illustrated in all the 65 algorithms written to solve different and historical probability problems in the following chapters.

Chapter III, defines the basic theory of probability and we will solve 29 important probability problems like the game of cards, the game of dice, the

game of domino, the game of letters, the game of chess, the game of coins, etc… and some historical problems like the birthday problem, De Meré problem, De Moivre problem, Bayes problem, Bernoulli problem, Huyghens problem, the principle of inclusion and exclusion, etc…that I discovered from my research and readings of many books on probability and statistics.

Chapter IV, deals with random variables. It includes four algorithms that illustrate the concept defined.

Chapter V, talks about mathematical expectation. It contains two algorithms as examples to the theory.

Chapter VI, applies the Monté Carlo technique to some well-known discrete and continuous probability and statistical distributions which are: The Binomial distribution, the continuous Chi-squared distribution, the Gamma and Exponential distributions, the *F*-distribution, the Geometric, the Hypergeometric, the Laplace, the Maxwell, the Negative Binomial distributions, the Standard Normal distribution, the Normal distribution, the Binomial versus the Poisson distribution, the *t*-distribution, the Bivariate Normal and the Cauchy distributions.

Chapter VII, studies random processes that are illustrated in three different algorithms about the random walk problems.

Chapter VIII, is an analysis of Markov chains and includes five algorithms illustrating the theory.

Chapter IX, is a development of Kolmogorov's axioms which are at the foundations of probability theory and hence it opens the door to a deterministic expression of probabilistic events. It is an interesting chapter indeed that should be read and that I preferred to include it in this present work like in my previous book *The Computer Simulation of Monté Carlo Methods and Random Phenomena* which was published in 2019 with Cambridge Scholars Publishing, since both books deal with random phenomena and probability theory. Five algorithms illustrate the original idea, but surely a whole dissertation can be written on this chapter alone, since determinism versus nondeterminism is a very deep debate among mathematicians and among physicists like between Albert Einstein and Niels Bohr…

Chapter X, finally, is a conclusion of the book. In the last chapter, we conclude this interesting and exciting topic with few pages in which we try to summarize the previous chapters developed in the manuscript.

I think that the topics chosen in the ten chapters of this book have served the purpose of illustrating probability theory and stochastic variables and processes. As a matter of fact, the subject is very broad and could be developed surely in many manuals and research papers… Thus, ten chapters are merely an introduction to this exciting, profound, and modern field of mathematics and knowledge.

CHAPTER II

FUNDAMENTAL MATHEMATICAL CONCEPTS AND METHODS

"*The known is finite, the unknown is infinite; intellectually we stand on an islet in the midst of an illimitable ocean of inexplicability. Our business in every generation is to reclaim a little more land.*"

Thomas Henry Huxley.

It is important before "probing the depths" of the stochastic paradigm, that we define some fundamental mathematical concepts and tools that will be extensively used in the whole manuscript algorithms. In fact, what follows is a list of some definitions and theorems that will applied in the subsequent chapters.

I- Simulation

Simulation is the process of designing a model of a real or imagined system and conducting experiments with this model to understand the behavior of the system or to evaluate strategies for its operation. Assumptions are made about this system and mathematical algorithms and relationships are derived to describe these assumptions – this constitutes a "model" that can reveal how the system works. If the system is simple, the model may be represented and solved analytically. A single equation such as DISTANCE = (RATE $\times$ TIME) may be an analytical solution representing the distance traveled by an object at a constant rate for a given period of time.

However, problems of interest in the real world are usually much more complex than this. In fact, they may be so complex that a closed analytical model cannot be constructed to represent them. In this case, the behavior of the system must be estimated through a simulation. Exact representation is seldom possible in a model, constraining us to approximations to a degree of fidelity that is acceptable for the purposes of the study. Models have been constructed for almost every system imaginable, including factories, communications and computer networks, integrated circuits, highway

systems, flight dynamics, national economics, social interactions, and imaginary worlds. In each of those environments, experimenting with a model of the system has proved to be more cost-effective, less dangerous, faster, or otherwise more practical than experimenting with a real system.

For example, a business may be interested in building a new factory to replace an old one, but is unsure whether the increased productivity will justify the investment. In this case, a simulation could be used to evaluate a model of the new factory. The model could describe the floor space required, the number of machines, the number of employees, the placement of equipment, the production capacity of each machine, and the waiting time between machines. The simulation runs would then evaluate the system and provide an estimate of the production capacity and the costs of a new factory. This type of information is invaluable in making decisions without having to build an actual factory to arrive at an answer.

Moreover, one of the pioneers of simulation was *John von Neumann*. In the late mid-1940s, together with physicist *Enrico Fermi* and mathematician *Stanislaw Ulam*, he conceived of the idea of running multiple repetitions of a model, gathering statistical data, and deriving behaviors of the real system based on these models. This came to be known as the Monté Carlo method because of the use of randomly generated variates to represent behaviors that could not be modeled exactly, but could be characterized statistically. Von Neumann used this method to study random actions of neutrons, which was simulated in Chapter V of my book *The Computer Simulation of Monté Carlo Methods and Random Phenomena*, and aircraft bombing effectiveness. Early civilian applications of this method were found in representations of factories attempting to determine maximum potential productivity.

Simulations derive much of their technique from models of the world found in other disciplines. Wind tunnels are models that replicate flight by moving the air rather than the aircraft; chess has been used to simulate strategic thinking about warfare; and computer games are intended to generate believable worlds requiring mastery of a specified set of behaviors.

II-The Monté Carlo Methods

The Computer Simulation of Monté Carlo Methods and Random Phenomena (pp. 1-2).

In applied mathematics, the name *Monté Carlo* is given to the method of solving problems by means of experiments with random numbers. This name, after the casino at Monaco, was first applied around 1944 to the method of solving deterministic problems by reformulating them in terms of a problem with random elements which could then be solved by large-scale sampling. But, by extension, the term has come to mean any simulation that uses random numbers.

The development and proliferation of computers has led to the widespread use of Monté Carlo methods in virtually all branches of science, ranging from nuclear physics (where computer-aided Monté Carlo was first applied) to astrophysics, biology, engineering, medicine, operations research, and the social sciences.

The Monté Carlo Method of solving problems by using random numbers in a computer – either by direct simulation of physical or statistical problems or by reformulating deterministic problems in terms of one incorporating randomness – has become one of the most important tools of applied mathematics and computer science. A significant proportion of articles in technical journals in such fields as physics, chemistry, and statistics contain articles reporting results of Monté Carlo simulations or suggestions on how they might be applied. Some journals are devoted almost entirely to Monté Carlo problems in their fields. Studies in the formation of the universe or of stars and their planetary systems use Monté Carlo techniques. Studies in genetics, the biochemistry of DNA, and the random configuration and knotting of biological molecules are studied by Monté Carlo methods. In number theory, Monté Carlo methods play an important role in determining primality or factoring of very large integers far beyond the range of deterministic methods. Several important new statistical techniques such as “bootstrapping” and “jackknifing” are based on Monté Carlo methods.

Hence, the role of Monté Carlo methods and simulation in all of the sciences has increased in importance during the past several years. These methods play a central role in the rapidly developing subdisciplines of the computational physical sciences, the computational life sciences, and the other computational sciences. Therefore, the growing power of computers and the evolving simulation methodology have led to the recognition of computation as a third approach for advancing the natural sciences, together with theory and traditional experimentation. At the kernel of Monté Carlo simulation is random number generation.

Now we turn to the approximation of a definite integral by the Monté Carlo method. If we select the first n elements $x_1, x_2, \ldots, x_n$ from a random sequence in the interval (0,1), then:

$$\int_0^1 f(x).dx \cong \frac{(1-0)}{n}\sum_{i=1}^{n} f(x_i) = \frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

Here the integral is approximated by the average of n numbers $f(x_1), f(x_2), \ldots, f(x_n)$. When this is actually carried out, the error is of order $\frac{1}{\sqrt{n}}$, which is not at all competitive with good algorithms, such as the *Romberg* method. However, in higher dimensions, the Monté Carlo method can be quite attractive. For example,

$$\int_0^1\int_0^1\int_0^1 f(x,y,z).dx.dy.dz \cong \frac{[(1-0)\times(1-0)\times(1-0)]}{n}\sum_{i=1}^{n} f(x_i,y_i,z_i)$$
$$= \frac{1}{n}\sum_{i=1}^{n} f(x_i,y_i,z_i)$$

where (x_i, y_i, z_i) is a random sequence of n points in the unit cube $0 \le x \le 1$ $0 \le y \le 1$, and $0 \le z \le 1$. To obtain random points in the cube, we assume that we have a random sequence in (0,1) denoted by $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \ldots$ To get our first random point p_1 in the cube, just let $p_1 = (\xi_1, \xi_2, \xi_3)$. The second is, of course, $p_2 = (\xi_4, \xi_5, \xi_6)$ and so on.

If the interval (in a one-dimensional integral) is not of length 1, but say is the general case (a, b), then the average of f over n random points in (a, b) is not simply an approximation for the integral but rather for:

$$\frac{1}{b-a}\int_a^b f(x).dx$$

which agrees with our intention that the function $f(x) = 1$ has an average of 1. Similarly, in higher dimensions, the average of f over a region is

obtained by integrating and dividing by the area, volume, or measure of that region. For instance,

$$\frac{1}{[(7-4)\times(5-(-2))\times(3-0)]}\int_4^7\int_{-2}^5\int_0^3 f(x,y,z).dx.dy.dz = \frac{1}{63}\int_4^7\int_{-2}^5\int_0^3 f(x,y,z).dx.dy.dz$$

is the average of f over the parallelepiped described by the following three inequalities:

$$0 \le x \le 3\,,\ -2 \le y \le 5\,,\ 4 \le z \le 7\,.$$

To keep the limits of integration straight, we recall that:

$$\int_a^b\int_c^d f(x,y).dx.dy = \int_a^b\left[\int_c^d f(x,y).dx\right].dy$$

and

$$\int_{a1}^{a2}\int_{b1}^{b2}\int_{c1}^{c2} f(x,y,z).dx.dy.dz = \int_{a1}^{a2}\left\{\int_{b1}^{b2}\left[\int_{c1}^{c2} f(x,y,z).dx\right].dy\right\}.dz$$

So, if (x_i, y_i) denote random points with appropriate uniform distribution, the following examples illustrate Monté Carlo techniques:

$$\int_1^9 f(x).dx \cong \frac{(9-1)}{n}\sum_{i=1}^n f(x_i) = \frac{8}{n}\sum_{i=1}^n f(x_i)$$

$$\int_4^8\int_2^5 f(x,y).dx.dy \cong \frac{[(8-4)\times(5-2)]}{n}\sum_{i=1}^n f(x_i,y_i) = \frac{12}{n}\sum_{i=1}^n f(x_i,y_i)$$

In each case, the random points should be uniformly distributed in the regions involved.

In general, we have:

$$\int_A f \cong (\text{measure of } A) \times (\text{average of } f \text{ over } n \text{ random points in } A)$$

Here we are using the fact that the average of a function on a set is equal to the integral of the function over the set divided by the measure of the set.

III- Random Numbers Generators

The generation of random numbers is also at the heart of many standard statistical methods. The random sampling required in most analysis is usually done by the computer. The computations required in Bayesian analysis have become viable because of Monté Carlo methods. This has led to much wider applications of Bayesian statistics, which, in turn, has led to the development of new Monté Carlo methods to the refinement of existing procedures for random numbers generation.

Various methods for the generation of random numbers have been used. Sometimes, processes that are considered random are used, but for Monté Carlo methods, which depend on millions of random numbers, a physical process as a source of random numbers is generally cumbersome. Instead of "random" numbers, most applications use "pseudorandom" numbers, which are deterministic but "look" like they were generated randomly. Chapter III of my book *The Computer Simulation of Monté Carlo Methods and Random Phenomena* discusses methods for the generation of sequences of pseudorandom numbers that simulate a uniform distribution over the interval (a,b). The book includes the basic sequences from which are derived pseudorandom numbers from other distributions, pseudorandom samples, and pseudo-stochastic processes.

Moreover, the Latin prefix "*pseudo*" means in English false, so when it is added to the word random, it describes clearly enough the process of generating random numbers. Why? This makes us return to the definition and the meaning of the word random that the philosophers have meditated upon. They said: is there any deterministic mathematical equation that could describe the inherent randomness existent in nature or is there none?… It is a philosophical debate that has never ended till our times. But for us, as computer scientists and mathematicians, we agree that we can write deterministic mathematical equations that can yield random numbers *nearly similar* to that existent in nature. Hence, we called the generators of those numbers: "*pseudorandom generators of random numbers"*.

In addition, the random number generator that we use in the present work is the C++ built-in generator that is called by the function *rand*(). The function *srand*() is also a C++ function that will generate a new random sequence on

each time we run the program. In fact, the latter takes its seed-value from the computer clock and hence each time the program is executed a new sequence is therefore produced. This surely helps in the analysis of a specific algorithm by giving us the chance to try the latter by a totally different and a new sequence of random numbers. The two C++ functions mentioned prove to be successful and efficient. Furthermore, like we have said, other random number generators exist in literature and they can be easily included in all the book algorithms to improve sometimes the accuracy of the programs results.

IV- Matrices

1- Definition

A matrix A is a rectangular array of numbers. An $m \times n$ matrix has m rows and n columns which is called the shape of the matrix. Such a matrix is denoted by $A = [a_{ij}]$ and is the following:

$$A(m,n) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

A matrix could be square when $m = n$. For example, if A is a square matrix of order 3 then it is the following:

$$A(m,m) = A(3,3) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

It can be also horizontal when $m = 1$ like:

$$A(1,n) = A(1,4) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix},$$

or vertical when $n = 1$ like: $A(m,1) = A(4,1) = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix}$.

Note that two matrices A and B are equal, if they have the same shape and the same corresponding entries. For example:

$$\text{If } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow A = B$$

$$\text{If } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \Rightarrow A \neq B$$

2- Matrix Addition and Subtraction

The addition and the multiplication of two matrices are possible if and only if the two matrices are of the same size. Then, $A+B$ is obtained by adding the corresponding entries of A and B, and $A - B$ is got by subtracting the corresponding elements of B from the elements of A.

For example:

$$A(3,3) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + B(3,3) = \begin{bmatrix} 4 & 5 & 6 \\ 6 & 7 & 8 \\ 9 & 10 & 11 \end{bmatrix} = C(3,3) = \begin{bmatrix} 5 & 7 & 9 \\ 10 & 12 & 14 \\ 16 & 18 & 20 \end{bmatrix}$$

And

$$A(3,3) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - B(3,3) = \begin{bmatrix} 4 & 5 & 6 \\ 6 & 7 & 8 \\ 9 & 10 & 11 \end{bmatrix} = C(3,3) = \begin{bmatrix} -3 & -3 & -3 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix}.$$

3- Scalar Multiplication

The matrix multiplication of A by a constant λ is $\lambda \times A$ and is obtained from A by multiplying all the entries of A by λ in this manner: