Quantum Gravity and Cosmology Based on Conformal Field Theory

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By

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PREFACE

Attempts to quantize gravity started from the 1960s and continue until now. However, in the framework of Einstein's theory, quantization of gravity did not go well because it does not become renormalizable in the standard way applied to other ordinary fields. Attempts were made to modify Einstein's theory to render it possible, but another difficulty known as ghost problem appeared, and eventually the attempt to quantize gravity with the standard field theory methods has gone away. After the 1980s, methods that do not depend on quantum field theory like string theory and loop quantum gravity have become mainstream. Many of books published in this research are about these theories. However, even though these theories have been studied for many years, realistic predictions that can explain briefly the current universe are not derived yet. Now is a good time to revisit the problem of quantization of gravity by returning to the traditional method again.

In this book, I will describe a renormalizable quantum gravity formulated with incorporating a new technique based on conformal field theory which recently has made prominent progress. Conformal invariance here appears as a gauge symmetry that gives a key property of quantum gravity known as the background-metric independence. Due to the presence of this symmetry, the theory becomes free from the problem of spacetime singularity, and thus from the information loss problem, namely the ghost problem as well. Furthermore, I will give a new scenario of the universe that evolves from quantum gravity world to the current classical world through the spacetime phase transition, including inflation driven by quantum gravity effects only.

This book also includes descriptions of recent developments in conformal field theory, renormalization theory in curved space, and conformal anomalies related thereto, almost of which do not found in other books. In addition, it includes review on evolution equations of the universe that is the foundation of modern cosmology necessary to understand results of the CMB experiments such as WMAP. Furthermore, it will be briefly shown that there is a noticeable relationship between the quantum gravity and a random lattice model that is based on the dynamical triangulation method known as another description of the background free property. I would like to describe these topics by taking enough pages as a latest advanced textbook for leading to this new area of quantum field theory that developed mainly since the beginning of this century.

Research results on the quantum gravity over the last twenty years are summarized in this book. I was helped by several collaborators in continuing this research. I am grateful to them for attending to my discussions for a long time. I especially thank Shinichi Horata and Tetsuyuki Yukawa. I could not proceed with this research without the help of two. This book is an English version of the book published in 2016 from the Pleiades Publishing in Japan, with adding a bit of new content and sentences. I appreciate the support of the Pleiades Publishing. In publishing this English version, I would like to thank Makoto Kobayashi, Kei-Ichi Kondo, and again Yukawa. I also wish to thank Cambridge Scholars Publishing for giving me the opportunity for publication. And I thank my wife Nonn and my daughter Kyouka for supporting me.

Tsukuba, Japan
June 2018

CHAPTER ONE

INTRODUCTION

The elementary particle picture represented by an ideal point without spreading is a concept incompatible with Einstein's theory of gravity.[1] Because such an object is nothing other than a black hole in terms of the theory of gravity. If its mass m is smaller than the Planck mass $m_{\rm pl}$, the Compton wavelength which gives a typical fluctuation size of particles becomes larger than the horizon size of the mass m, hence it can be approximated as a particle. However, in the world beyond the Planck scale, such an approximation does not hold because particle information is confined inside the horizon (see Fig. 1-1).

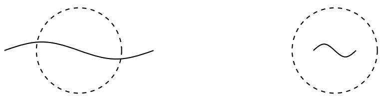

Figure 1-1: The Compton wavelength of mass m is given by $\lambda \sim 1/m$, while particle's horizon size (dotted line) is $r_g \sim m/m_{\rm pl}^2$. Therefore, $\lambda < r_g$ for $m > m_{\rm pl}$, as shown on the right, and information on such an elementary excitation is confined inside the horizon and lost. Hence, in the world beyond the Planck scale, normal particle picture is no longer established.

The goal of quantum gravity is to reveal a high energy physics beyond the Planck scale. While particles live in spacetime, gravity rules the spacetime itself, and the difference between their roles stands out there. Quantum fluctuations of gravity become large, so that the concept of time and distance will be lost. Quantization of the spacetime itself is required to describe such a world where the image of particles moving in a specific spacetime is broken.

One way to resolve the problem mentioned above is to realize such a

[1] The original paper is A. Einstein, *Die Grundlage der allgemeinen Relativitätstheorie*, Annalen der Phys. **49** (1916) 769.

quantum spacetime where the scale itself does not exist. It can be represented as gauge equivalence between spacetimes with different scales. This property is called the background-metric independence. In this book, as a theory with such a property, we will present a renormalizable quantum field theory, called the asymptotically background-free quantum gravity, whose ultraviolet limit is described as a special conformal field theory that has conformal invariance as a gauge symmetry.

Academic Interests

From observations of the cosmic microwave background (CMB) radiation by Wilkinson Microwave Anisotropies Probe (WMAP), which is an astronomical satellite launched from the NASA Kennedy Space Center in 2001, cosmological parameters were determined with high precision and the theory of inflation which suggests that a rapid expansion occurred in the early stage of the universe was strongly supported. On the other hand, there are still many simple and fundamental questions left, for example, why the universe is expanding or what is the source of repulsive force that ignites inflation.

Interpreting the inflation theory naturally, the universe has expanded about 10^{60} times from the birth to the present. This means that the larger size than a cluster of galaxies was within the Planck length $l_{\rm pl}$ before inflation begins. It suggests that traces of quantum fluctuations of gravity in the creation period of the universe are recorded in the CMB anisotropy spectrum observed by WMAP.

Cosmic expansion, the big bang, creation of the primordial fluctuations, and so on, it seems to be natural to consider that their origin is in quantum effects of gravity. Quantum gravity is expected as a necessary physics to understand the history of the universe from the birth of spacetime to the present. The ultimate goal of this book is to explain the spectrum of CMB using the asymptotically background-free quantum gravity. Recent studies have revealed that we can explain a number of observed facts well if considering that a spacetime phase transition suggested by this theory as the big bang occurred at 10^{17}GeV.

Historical Background

Einstein's theory of gravity has many properties unfavorable in constructing its quantum theory, for examples, the Einstein-Hilbert action given by the

Ricci scalar curvature is not positive-definite, and the Newton coupling constant has dimensions so that the theory becomes unrenormalizable. However, renormalization itself is not an idea contradictory to diffeomorphism invariance, or invariance under general coordinate transformations, which is the basis of the gravity theory.

In the early studies of the 1970s, it was considered that renormalizable quantum gravity could be obtained by simply adding fourth-order derivative gravitational actions to the Einstein-Hilbert action. It is because due to the fact that the gravitational field is dimensionless unlike other known fields, not only the coupling constant becomes dimensionless, but also the action can be made positive-definite. Furthermore, when including the Riemann curvature tensor in the action, spacetime singularities can be removed quantum mechanically because the action diverges for such field configurations.

However, with methods of treating all modes of the gravitational field perturbatively, we could not prevent undesirable gauge-invariant ghosts from appearing as asymptotic fields. It is the problem of the so-called massive graviton with negative metric.[2] Eventually, the attempt to quantize gravity with standard methods of quantum field theory had gone away, and after the 1980s, methods that do not use quantum field theory have become mainstream. Actually, there are many studies on quantum gravity, but there are few ones that have directly performed quantization of the gravitational field.

The purpose of this book is to return to the traditional method of quantum field theory again and propose a new approach to renormalizable quantum theory of gravity. In order to solve the problems, we introduce a nonperturbative technique based on conformal field theory which has recently made remarkable progress. As the result, the particle picture propagating in a specific background will be discarded.

A significant progress in methods to quantize gravity was made in the latter half of the 1980s. That is the discovery of an exact solution of two-dimensional quantum gravity. The major difference from the conventional quantum gravity mainly studied from the 1970s to the early 1980s was that it correctly took in contributions from the path integral measure and treated the conformal factor in the metric tensor field strictly. This study indicated

---

[2] There is a work on the unitarity issue by T. Lee and G. Wick, Nucl. Phys, **B9** (1969) 209, in which they proposed an idea that considering a full propagator including quantum corrections, a real pole representing the existence of ghosts disappears and moves to a pair of complex poles so that ghosts do not appear in the real world. For its application to quantum gravity, see E. Tomboulis, Phys. Lett. **70B** (1977) 361 and references in Bibliography. For more detailed explanations, see the end of Chapter 7. However, this idea cannot be applied to the ultraviolet limit where interactions turn off.

that diffeomorphism invariance in quantum theory should be handled more carefully than that in classical theory.

The essence of this approach is that diffeomorphism invariance involves conformal invariance, thus the quantum gravity theory is formulated as a certain conformal field theory defined on any background spacetime. The difference from normal conformal field theory is that conformal invariance is a gauge symmetry, namely BRST symmetry.[3] In normal conformal field theory, only the vacuum is conformally invariant, whereas in the quantum gravity fields must be conformally invariant as well. All of the theories with different backgrounds connected each other by conformal transformations become gauge-equivalent, thus the background-metric independence is realized. This is called the BRST conformal invariance. It represents that the so-called Wheeler-DeWitt algebra is realized at the quantum level.[4]

Developing this method in four dimensions, we have formulated a new renormalizable quantum theory of gravity. The gravitational field is then decomposed into three parts: the conformal factor defined in an exponential, the traceless tensor field, and a background metric. By quantizing the conformal factor in a non-perturbative way, the background-metric independence is strictly realized as the BRST conformal invariance in the ultraviolet limit. On the other hand, dynamics of the traceless tensor field which cannot be ignored in four dimensions is handled perturbatively by adding the fourth-order derivative Weyl action. Since the coupling constant becomes dimensionless, the theory becomes renormalizable.

In conventional quantum field theories based on Einstein's theory of gravity, the Planck scale is usually regarded as an ultraviolet cutoff. Hence, problems of spacetime singularities, ultraviolet divergences, and even the cosmological constant are substantially avoided. On the other hand, this new renormalizable quantum gravity does not require such an ultraviolet cutoff, because the beta function of the gravitational coupling constant becomes negative, like in quantum chromodynamics (QCD). Therefore, we

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[3] BRST is an abbreviation for Becchi-Rouet-Stora-Tyutin which arranged the names of four discovers. The original papers are C. Becchi, A. Rouet, and R. Stora, *Renormalization of the Abelian Higgs-Kibble Model*, Comm. Math. Phys. **42** (1975) 127; *Renormalization of Gauge Theories*, Ann. Phys. **98** (1976) 287, and I. Tyutin, Lebedev preprint FIAN, 1975. See T. Kugo and I. Ojima, *Local Covariant Operator Formalism of Non-Abelian Gauge Theories and Quark Confinement Problem*, Prog. Theor. Phys. Suppl. **66** (1979) 1 and reference books on quantum field theory in Bigliography.

[4] At the classical Poisson bracket level, the Wheeler-DeWitt algebra holds for arbitrary diffeomorphism invariant theory, but for the algebra to close at the quantum level, the theory is constrained, so that the gravitational action is determined tightly.

can describe a world beyond the Planck scale.

Furthermore, the massive graviton mode becomes unphysical in this approach, because a quadratic term of the field giving mass to this mode is not gauge invariant due to the existence of the exponential conformal factor in the Einstein-Hilbert action. Not only that, the BRST conformal symmetry shows that all modes in the fourth-order derivative gravitational field are not gauge invariant after all even in the ultraviolet limit.

As a theoretical background in which this four-dimensional quantum gravity was devised, there is a work of numerical calculations by the dynamical triangulation method.[5] It is a random lattice model in which the two-dimensional model (matrix model) is generalized to four dimensions, and the simulation result strongly suggested that scalar fluctuations are more dominant than tensor fluctuations. From this research result, we came up with this quantization method which treats only the traceless tensor field perturbatively.

After that, the first observation result of WMAP was released in 2003, and it was indicated that a scale-invariant scalar fluctuation dominates in the early universe. At the same time, the existence of a new scale close to the Planck length was suggested. At first, we could not imagine that a wavelength of observed fluctuations about 5000Mpc which corresponds to the size of the universe is related with the smallest length scale among the known ones, but it can be understood when we consider that the universe expanded about 10^{60} times from its birth to the present, including an inflationary period and the subsequent 13.7 billion years, predicted from a typical scenario of the inflation theory. From the consideration of this new scale, the idea of quantum gravity inflation was born.

Excellent Points of The Theory

A theoretical superiority of the BRST conformal field theory is that whatever background metric we choose, as far as it is conformally flat, the theory does not lose its generality. With this theory as the core, the renormalizable quantum gravity can be constructed as a quantum field theory in the flat background as usual. Dynamics that represents a deviation from the conformal invariance is controlled by only one dimensionless gravitational

[5] See S. Horata, H. Egawa, and T. Yukawa, *Clear Evidence of A Continuum Theory of 4D Euclidean Simplicial Quantum Gravity*, Nucl. Phys. B (Proc. Suppl.) **106** (2002) 971; S. Horata, H. Egawa, and T. Yukawa, *Grand Canonical Simulation of 4D Simplicial Quantum Gravity* , Nucl. Phys. B (Proc. Suppl.) **119** (2003) 921. See also the fifth section of Appendix D and the author's review article in Bibliography.

coupling constant whose beta function becomes negative.

The renormalization theory is formulated using dimensional regularization, which is a regularization method that can calculate higher loop quantum corrections while preserving diffeomorphism invariance. The long-standing problem that the form of fourth-order gravitational actions cannot be fixed from classical diffeomorphism invariance alone is settled at the quantum level, that is, it is determined by not only imposing the Wess-Zumino integrability condition but also using a certain new renormalization group equation.

The fact that the beta function is negative means that the theory can be defined correctly in the ultraviolet limit. Unlike conventional quantum field theory, however, it does not indicate that the flat spacetime in which asymptotic fields can be defined is realized. This is because the conformal factor still fluctuates non-perturbatively so that spacetime is fully quantum mechanical. Therefore, the traditional S-matrix is not defined as a physical quantity. In this book, we refer to this behavior as "asymptotic background freedom", in distinction from the conventional asymptotic freedom.

It also suggests the existence of a new dynamical infrared energy scale of quantum gravity denoted by $\Lambda_{\rm QG}$ here, like $\Lambda_{\rm QCD}$ in QCD.[6] At sufficiently high energy beyond $\Lambda_{\rm QG}$, tensor fluctuations become smaller, while scalar fluctuations by the conformal factor dominate. Below $\Lambda_{\rm QG}$, such conformal dynamics disappears. Thus, this scale divides quantum spacetime filled with conformal fluctuations of gravity from the current classical spacetime without conformal invariance. The more detailed physical implications indicated by this scale are as follows.

Inflation and spacetime phase transition If setting the magnitude relation between the Planck mass $m_{\rm pl}$ and the dynamical scale $\Lambda_{\rm QG}$ as $m_{\rm pl} > \Lambda_{\rm QG}$, there is an inflationary solution, then evolution of the early universe can be divided into three eras separated by these two scales. At high energy far beyond the Planck scale it is described as a conformally invariant spacetime where quantum scalar fluctuations of the conformal factor dominate. The conformal invariance starts breaking in the vicinity of the Planck scale, and gradually shifts to the era of inflation. The inflationary era drastically ends at $\Lambda_{\rm QG}$ where the conformal invariance loses its validity completely. At this point, the universe is expected to make a transition

[6] The existence of such a scale is a characteristic of renormalizable quantum field theory, which is a scale that does not exist in a manifestly finite continuum theory like string theory. It is also characterized by the fact that the effective action has a nonlocal form, and this point is also different from a manifestly finite theory which generally gives a local effective theory.

to the classical Friedmann spacetime in which long distance correlation has been lost. If we choose this scale as 10^{17}GeV, we can explain the CMB observation results well.

One of the excellent points in this inflationary scenario is that it can explain the evolution of the universe using the dynamics of the gravitational field alone without introducing a phenomenological scalar degree of freedom called the inflaton.[7] Interactions between the conformal-factor field and matter fields open through conformal anomaly, and become strong rapidly near $\Lambda_{\rm QG}$. The big bang is caused by that a fourth-order derivative scalar degree of freedom in the conformal factor changes to matter fields immediately at the time of the spacetime phase transition. Hence, it is suggested that quantum fluctuations of gravity are the source of everything. The origin of primordial fluctuations necessary for explaining the structure formation of the universe is given by a scale-invariant scalar spectrum predicted from conformal invariance.

Existence of physical minimum length The dynamical scale $\Lambda_{\rm QG}$ separating quantum and classical spacetimes implies that there is no concept of distance shorter than the correlation length $\xi_\Lambda = 1/\Lambda_{\rm QG}$ because spacetime totally fluctuates there. In this sense, ξ_Λ denotes a minimum length we can measure. Thus, spacetime is practically quantized by ξ_Λ, without discretizing it explicitly, that is, without breaking diffeomorphism invariance. Excitations in quantum gravity would be given by the mass of order $\Lambda_{\rm QG}$.

Although we do not know how large our universe is, at least most of the range that we are looking at today falls within the minimum length before inflation, because the present Hubble distance is given by the order of $10^{59} \times \xi_\Lambda$, as mentioned before. That is to say, we can consider that the universe we are observing now was born from a “bubble” of quantum gravity fluctuations. This is the reason why the primordial spectrum of the universe is almost scale invariant.

On the other hand, since correlations larger than ξ_Λ disappear, the sharp fall-off observed in large angular components of the CMB anisotropy spectrum can be explained by this length scale.

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[7] On the other hand, Einstein’s theory of gravity is a theory that matter density determines the structure of spacetime. In other words, the current spacetime cannot be produced from the absence of matters. Therefore, the inflation model based on Einstein’s theory of gravity has to introduce a scalar field as a source of all matter fields, but it is unconvincing that elementary particles with a theoretical background such as gauge principles and renormalizability are created from a scalar field that does not have these properties.

New approach to unitarity problem As stated at the beginning, in gravity theories based on the Einstein-Hilbert action, an elementary excitation that has energy beyond the Planck mass becomes a black hole, and thus unitarity is broken. On the other hand, the asymptotically background-free theory indicates that spacetime configurations where the Weyl curvature tensor disappears dominate at high energy beyond $\Lambda_{\rm QG}$. Therefore, spacetime configuration where the Riemann curvature tensor diverges like the Schwarzschild solution is excluded at the quantum level.[8] The existence of such a singular point is also denied by the realization of the BRST conformal invariance representing the background-metric independence.

Since singularities are eliminated, it is possible to discuss the problem of unitarity non-perturbatively. Algebraically, conformal invariance becomes important. The unitarity in conformal field theory is that the Hermitian nature of fields is preserved even in correlation functions. It is expressed as the conditions that not only two-point functions are positive-definite but also operator product expansion coefficients are real.[9]

The BRST conformal invariance gives far stronger constraints on the theory than conventional conformal invariance. Negative-metric ghost modes included in the fourth-order derivative gravitational field are necessary for the conformal algebra to close, but they are not gauge invariant themselves, so that they do not appear in the real world. Physical operators are given by real primary scalar composite fields with a specific conformal dimension, whereas fields with tensor indices become unphysical. Since the whole action is positive-definite, the stability of the path integral is guaranteed, and thus the Hermitian nature of the physical operators will be retained.[10]

---

[8] Since the Weyl action diverges, singular configurations are obviously unphysical, whereas in Einstein's gravity theory such a singularity cannot be eliminated because the Einstein-Hilbert action given by the Ricci scalar vanishes, that is, it is physical.

[9] Since the action is often unknown in conformal field theory, such conditions will be imposed (see Chapters 2 and 3). If the action is known, it can be easily understood in statistical mechanics by considering Wick-rotated Euclidean space. If the Euclidean action I is positive-definite, the path integral with weight e^{-I} is correctly defined and thus reality of fields is preserved. If the action is not bounded from below, the path integral diverges and thus the field reality is sacrificed in order to regularize it.

[10] Each mode in the fourth-order gravitational field is not a physical quantity, thus as long as considering correlation functions of physical fields, the positivity of the whole action expressed by the original gravitational field is essential (see Footnote 9).

Outline of The Book

In Chapters 2 and 3, we explain the basis of conformal field theory and recent developments. The basis of two-dimensional conformal field theory is summarized in Chapter 4. In Chapter 5, we describe conformal anomaly involved deeply in the construction of quantum gravity. Chapter 6 is devoted to two-dimensional quantum gravity, which is the simplest theory with the BRST conformal invariance.

In Chapters 7 and 8, we formulate the BRST conformally invariant quantum gravity in four dimensions, which is one of the main subjects of this book, and construct physical field operators and physical states. As a first step to define renormalizable quantum theory of gravity by using dimensional regularization, we examine quantum field theory in curved spacetime in Chapter 9. The form of gravitational counterterms and conformal anomalies is then determined using an advanced technique of renormalization group equations applied to composite fields. Based on this result, we formulate the renormalizable asymptotically background-free quantum gravity in Chapter 10.

In the last four chapters we will discuss evolution of the universe that the quantum gravity suggests. In order to show why we can consider that its traces remains today, we first explain the Friedmann universe in Chapter 11, then present a model of inflation induced by quantum gravity effects in Chapter 12. Furthermore, in Chapter 13, we explain in detail cosmological perturbation theory describing time evolution of fluctuations. In Chapter 14, we apply it to the quantum gravity cosmology and examine time evolution of quantum gravity fluctuations in the inflationary background, then show that the amplitudes reduce during inflation. From quantum gravity spectra given before the Planck time, we derive primordial power spectra right after the spacetime phase transition, and with them as initial spectra of the Friedmann universe, the CMB anisotropy spectra are calculated and compared with experimental data.

Each chapter of the appendix supplements useful formulas for gravitational fields and also useful knowledge that will help understanding although it is slightly out of the main subject.

Finally, from the author's review article listed in Bibliography, extract the following passage:

> The wall of Planck scale reminds us the wall of sound speed. When an airplane speeds up and approaches to the sound speed, it faces violent vibrations due to the sound made by the airplane itself and sometimes breaks the airplane into pieces. People of old days thought that the sound speed is the

> unpassable wall. However, we know now once we pass the wall with durable body, a peaceful space without sounds spreads about us. Similarly, we might think that the Planck scale is the wall that we can never pass. However, once we go beyond the Planck scale, there is no singularity, but a harmonious space of conformal symmetry.

The thought of this research is summarized in this sentence.

Chapter Two

Conformal Field Theory in Minkowski Space

As places where conformal field theory appears, non-trivial fixed points in quantum field theories and critical points in statistical models are widely known. In addition, we will show that an ultraviolet limit of quantum gravity is described as a certain conformal field theory in this book.

These theories will be discussed in Minkowski space or Euclidean space, and each has advantages. First of all, the basis of conformal field theory in Minkowski space are summarized. In this case, the procedure of quantization, the Hamiltonian operator, the nature of field operators such as Hermiticity, etc. are more clear than quantum field theory in Euclidean space. Conformal field theory in Euclidean space is basically considered to be obtained by analytic continuation from Minkowski space.

On the other hand, in the case where an action or a (non-perturbative) quantization method is not clear, it is easier to discuss in Euclidean space, because we can avoid divergences specific to Minkowski space. In addition, there are advantages such as structures of correlation functions, correspondences between states and operators, and so on become clearer, and also correspondences with statistical mechanics becomes easy to understand. Conformal field theory in Euclidean space is discussed in the next chapter.

Hereinafter, when describing the basic properties of conformal field theory, we describe them in any D dimensions. When presenting specific examples, calculations are done in four dimensions for simplicity.

Conformal Transformations

Conformal transformations are coordinate transformations in which when transforming coordinates to $x^\mu \to x'^\mu$, a line element changes as

$$\eta_{\mu\nu} dx^\mu dx^\nu \to \eta_{\mu\nu} dx'^\mu dx'^\nu = \Omega^2(x) \eta_{\mu\nu} dx^\mu dx^\nu, \qquad (2\text{-}1)$$

where Ω is an arbitrary real function and the Minkowski metric is $\eta_{\mu\nu} = (-1, 1, \cdots, 1)$. Rewriting the right-hand side, the conformal transformation is expressed as

$$\eta_{\mu\nu}\frac{\partial x'^{\mu}}{\partial x^{\lambda}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}} = \Omega^2(x)\eta_{\lambda\sigma}.$$

The $\Omega = 1$ case corresponds to the Poincaré transformation.

Conformal transformations are defined only on the background metric $\eta_{\mu\nu}$, and under the transformation this metric tensor itself does not change. On the other hand, diffeomorphism is a coordinate transformation in which the metric tensor is regarded as a field to transform together in order to preserve the line element as a scalar quantity, thus it has to be distinguished from the conformal transformation.[1] Below, all contractions of indices of tensor fields are done with the background metric $\eta_{\mu\nu}$.

Considering an infinitesimal conformal transformation $x^\mu \to x'^\mu = x^\mu + \zeta^\mu$, we find from the above equation that ζ^μ must satisfy

$$\partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu - \frac{2}{D}\eta_{\mu\nu}\partial_\lambda \zeta^\lambda = 0. \tag{2-2}$$

This is called the conformal Killing equation, and ζ^λ is called the conformal Killing vector. The arbitrary function is then given by

$$\Omega^2 = 1 + \frac{2}{D}\partial_\lambda \zeta^\lambda. \tag{2-3}$$

Deforming the conformal Killing equation (2-2), we get

$$\left[\eta_{\mu\nu}\partial^2 + (D-2)\partial_\mu\partial_\nu\right]\partial_\lambda\zeta^\lambda = 0.$$

Furthermore, since $(D-1)\partial^2\partial_\lambda\zeta^\lambda = 0$ is obtained from the trace of this expression, we get $\partial_\mu\partial_\nu\partial_\lambda\zeta^\lambda = 0$ for $D > 2$.[2] Solving the equation with this in mind yields $(D+1)(D+2)/2$ solutions. They correspond to D translations, $D(D-1)/2$ Lorentz transformations, one dilatation, D special conformal transformations, denoted by $\zeta^\lambda_{T,L,D,S}$, respectively, which are given as follows:

$$\begin{aligned} (\zeta^\lambda_T)_\mu &= \delta^\lambda_{\ \mu}, \qquad (\zeta^\lambda_L)_{\mu\nu} = x_\mu\delta^\lambda_{\ \nu} - x_\nu\delta^\lambda_{\ \mu}, \\ \zeta^\lambda_D &= x^\lambda, \qquad (\zeta^\lambda_S)_\mu = x^2\delta^\lambda_{\ \mu} - 2x_\mu x^\lambda. \end{aligned} \tag{2-4}$$

[1] Considered the metric as a field and combined with diffeomorphism, the conformal transformation can be expressed as the Weyl rescaling of the metric field, but in this and the next two chapters it is not considered.

[2] In $D = 2$, the condition reduces to $\partial^2\partial_\lambda\zeta^\lambda = 0$, and the number of the conformal Killing vectors becomes infinite.

The indices μ, ν here represent the degrees of freedom of $\zeta^\lambda_{T,L,S}$. The first two correspond to isometry transformations that satisfies the Killing equation $\partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu = 0$, namely the Poincaré transformations.

Finite conformal transformations for dilatation and special conformal transformation are given by

$$x^\mu \to x'^\mu = \lambda x^\mu, \qquad x^\mu \to x'^\mu = \frac{x^\mu + a^\mu x^2}{1 + 2a_\mu x^\mu + a^2 x^2},$$

respectively. In addition to these, we introduce conformal inversion

$$x^\mu \to x'^\mu = \frac{x^\mu}{x^2}, \tag{2-5}$$

which is an important transformation that can be used in place of special conformal transformation. By combining conformal inversion and translation, special conformal transformation can be derived as

$$x^\mu \to \frac{x^\mu}{x^2} \to \frac{x^\mu}{x^2} + a^\mu \to \frac{\frac{x^\mu}{x^2} + a^\mu}{\left(\frac{x^\mu}{x^2} + a^\mu\right)^2} = \frac{x^\mu + a^\mu x^2}{1 + 2a_\mu x^\mu + a^2 x^2}.$$

Conformal Algebra and Field Transformation Law

Let P_μ, $M_{\mu\nu}$, D, and K_μ be generators of translation, Lorentz transformation, dilatation and special conformal transformation, respectively.[3] These $(D+1)(D+2)/2$ infinitesimal conformal transformation generators satisfy the following $SO(D,2)$ algebra:[4]

$$\begin{aligned}
&[P_\mu, P_\nu] = 0, \qquad [M_{\mu\nu}, P_\lambda] = -i\left(\eta_{\mu\lambda} P_\nu - \eta_{\nu\lambda} P_\mu\right), \\
&[M_{\mu\nu}, M_{\lambda\sigma}] = -i\left(\eta_{\mu\lambda} M_{\nu\sigma} + \eta_{\nu\sigma} M_{\mu\lambda} - \eta_{\mu\sigma} M_{\nu\lambda} - \eta_{\nu\lambda} M_{\mu\sigma}\right), \\
&[D, P_\mu] = -iP_\mu, \quad [D, M_{\mu\nu}] = 0, \quad [D, K_\mu] = iK_\mu, \\
&[M_{\mu\nu}, K_\lambda] = -i\left(\eta_{\mu\lambda} K_\nu - \eta_{\nu\lambda} K_\mu\right), \quad [K_\mu, K_\nu] = 0, \\
&[K_\mu, P_\nu] = 2i\left(\eta_{\mu\nu} D + M_{\mu\nu}\right).
\end{aligned} \tag{2-6}$$

[3] In this book, the same symbol D as spacetime dimensions is used for the generator of dilatation. They can be readily distinguished from the context.

[4] In two dimensions, the $SO(2,2)$ conformal algebra is extended to the infinite dimensional Virasoro algebra and what is called the central charge appears, but such a central extension does not exist in the conformal algebra of $D > 2$.

A subalgebra $SO(D-1,1)$ composed of the generators of translation and Lorentz transformation is the Poincaré algebra. Hermiticity of the generators is defined by

$$P_\mu^\dagger = P_\mu, \quad M_{\mu\nu}^\dagger = M_{\mu\nu}, \quad D^\dagger = D, \quad K_\mu^\dagger = K_\mu.$$

The conformal algebra can be represented collectively using the generator of $SO(D,2)$ denoted by J_{ab} as

$$[J_{ab}, J_{cd}] = -i\left(\eta_{ac}J_{bd} + \eta_{bd}J_{ac} - \eta_{bc}J_{ad} - \eta_{ad}J_{bc}\right), \tag{2-7}$$

where the metric is set to be $\eta_{ab} = (-1,1,\ldots,1,-1)$, numbering as $a, b = 0,1,2,\ldots,D,D+1$. The generator is antisymmetric $J_{ab} = -J_{ba}$ and satisfies Hermiticity $J_{ab}^\dagger = J_{ab}$. Indeed, the conformal algebra (2-6) is obtained by choosing the spacetime indices as $\mu, \nu = 0,1,\ldots,D-1$ and writing the generators as

$$M_{\mu\nu} = J_{\mu\nu}, \qquad D = J_{D+1D},$$
$$P_\mu = J_{\mu D+1} - J_{\mu D}, \qquad K_\mu = J_{\mu D+1} + J_{\mu D}.$$

Fields that transform regularly under conformal transformations are particularly called primary fields. We here consider a symmetric traceless tensor field $O_{\mu_1\cdots\mu_l}$ representing a field of integer spin l.[5] Let Δ be a conformal dimension and the field satisfies Hermiticity

$$O_{\mu_1\cdots\mu_l}^\dagger(x) = O_{\mu_1\cdots\mu_l}(x).$$

A primary scalar field is defined so that it transforms under conformal transformations as

$$O'(x') = \Omega^{-\Delta}(x)O(x).$$

Since $O_{\mu_1\cdots\mu_l}(x)dx^{\mu_1}\cdots dx^{\mu_l}$ transforms as a scalar quantity of conformal dimension $\Delta - l$, the transformation law of a primary tensor field is then given by

$$O'_{\mu_1\cdots\mu_l}(x') = \Omega^{l-\Delta}(x)\frac{\partial x^{\nu_1}}{\partial x'^{\mu_1}}\cdots\frac{\partial x^{\nu_l}}{\partial x'^{\mu_l}}O_{\nu_1\cdots\nu_l}(x). \tag{2-8}$$

[5] In $D = 4$, this is a tensor field corresponding to the $j = \tilde{j} = l/2$ case in the $(j,\tilde{j})$ representation of the Lorentz group $SO(3,1)$, which can be expressed as $O_{\mu_1\cdots\mu_l} = (\sigma_{\mu_1})^{\alpha_1\dot{\alpha}_1}\cdots(\sigma_{\mu_l})^{\alpha_l\dot{\alpha}_l}O_{\alpha_1\cdots\alpha_l\dot{\alpha}_1\cdots\dot{\alpha}_l}$. In addition, as fields with $j \neq \tilde{j}$, spinor fields of $(1/2,0)$ and $(0,1/2)$, Rarita-Schwinger fields of $(1,1/2)$ and $(1/2,1)$, antisymmetric tensor fields of $(1,0)$ and $(0,1)$, and so on are widely known.

Denoting a vector representation of the orthogonal group $SO(D-1,1)$ as $D_{\mu\nu}$, Jacobian of the transformation is decomposed in the form $\partial x^\nu/\partial x'^\mu = \Omega^{-1}(x) D_\mu^{\ \nu}(x)$. Here, a primary field of arbitrary spin is simply denoted as $O_j(x)$ and the representation matrix acting on it is written as $R[D]_{jk}$. The conformal transformation can then be expressed with a combination of scale transformations and rotations as $O'_j(x') = \Omega^{-\Delta}(x) R[D(x)]_j^{\ k} O_k(x)$.

If the vacuum $|0\rangle$ is conformally invariant, correlation functions of these operators satisfy

$$\langle 0|O_{j_1}(x_1)\cdots O_{j_n}(x_n)|0\rangle = \langle 0|O'_{j_1}(x_1)\cdots O'_{j_n}(x_n)|0\rangle, \qquad \text{(2-9)}$$

where note that the argument of the field on the right-hand side is x_j, which is the same as the left-hand side.

The conformal transformation law under an infinitesimal change $x^\mu \to x'^\mu = x^\mu + \zeta^\mu$ is derived by expanding $\delta_\zeta O_j(x) \equiv O_j(x) - O'_j(x)$ by ζ^μ. Noting that $O'_j(x' = x+\zeta) = O'_j(x) + \zeta^\mu \partial_\mu O_j(x)$, $D_\nu^{\ \mu} = \delta_\nu^{\ \mu} - (\partial_\nu \zeta^\mu - \partial^\mu \zeta_\nu)/2$, and (2-3), an infinitesimal conformal transformation of primary tensor fields is given by

$$\begin{aligned}\delta_\zeta O_{\mu_1\cdots\mu_l}(x) &= \left(\zeta^\lambda \partial_\lambda + \frac{\Delta}{D}\partial_\lambda \zeta^\lambda\right) O_{\mu_1\cdots\mu_l}(x) \\ &\quad + \frac{1}{2}\sum_{j=1}^{l}\left(\partial_{\mu_j}\zeta^\lambda - \partial^\lambda \zeta_{\mu_j}\right) O_{\mu_1\cdots\mu_{j-1}\lambda\,\mu_{j+1}\cdots\mu_l}(x)\end{aligned}$$

from the transformation law (2-8).

The infinitesimal transformation is expressed as a commutator between the generator and the field operator as

$$\delta_\zeta O_{\mu_1\cdots\mu_l}(x) = i\left[Q_\zeta, O_{\mu_1\cdots\mu_l}(x)\right],$$

where Q_ζ is a generic name of $(D+1)(D+2)/2$ generators for the conformal Killing vector ζ^λ. By substituting the concrete forms of the conformal Killing vectors $\zeta^\lambda_{T,L,D,S}$ (2-4), we obtain the following transformation laws:

$$\begin{aligned} i\left[P_\mu, O_{\lambda_1\cdots\lambda_l}(x)\right] &= \partial_\mu O_{\lambda_1\cdots\lambda_l}(x), \\ i\left[M_{\mu\nu}, O_{\lambda_1\cdots\lambda_l}(x)\right] &= \left(x_\mu\partial_\nu - x_\nu\partial_\mu - i\Sigma_{\mu\nu}\right) O_{\lambda_1\cdots\lambda_l}(x), \\ i\left[D, O_{\lambda_1\cdots\lambda_l}(x)\right] &= \left(x^\mu\partial_\mu + \Delta\right) O_{\lambda_1\cdots\lambda_l}(x), \\ i\left[K_\mu, O_{\lambda_1\cdots\lambda_l}(x)\right] &= \left(x^2\partial_\mu - 2x_\mu x^\nu \partial_\nu - 2\Delta x_\mu + 2ix^\nu\Sigma_{\mu\nu}\right) O_{\lambda_1\cdots\lambda_l}(x), \end{aligned} \qquad \text{(2-10)}$$

where spin term is defined by

$$\Sigma_{\mu\nu} O_{\lambda_1 \cdots \lambda_l} = i \sum_{j=1}^{l} \left(\eta_{\mu\lambda_j} \delta^{\sigma}_{\ \nu} - \eta_{\nu\lambda_j} \delta^{\sigma}_{\ \mu} \right) O_{\lambda_1 \cdots \lambda_{j-1} \sigma \lambda_{j+1} \cdots \lambda_l}.$$

If defining a spin matrix as $\Sigma_{\mu\nu} O_{\lambda_1 \cdots \lambda_l} = (\Sigma_{\mu\nu})_{\lambda_1 \cdots \lambda_l}^{\ \ \sigma_1 \cdots \sigma_l} O_{\sigma_1 \cdots \sigma_l}$, then it satisfies the same algebra as the Lorentz generator $M_{\mu\nu}$. In the case of a vector field, it is given by $(\Sigma_{\mu\nu})_{\lambda}^{\ \sigma} = i(\eta_{\mu\lambda} \delta_{\nu}^{\ \sigma} - \eta_{\nu\lambda} \delta_{\mu}^{\ \sigma})$, and the general formula of l is represented using it as

$$(\Sigma_{\mu\nu})_{\lambda_1 \cdots \lambda_l}^{\ \ \sigma_1 \cdots \sigma_l} = \sum_{j=1}^{l} \delta_{\lambda_1}^{\ \sigma_1} \cdots \delta_{\lambda_{j-1}}^{\ \sigma_{j-1}} (\Sigma_{\mu\nu})_{\lambda_j}^{\ \sigma_j} \delta_{\lambda_{j+1}}^{\ \sigma_{j+1}} \cdots \delta_{\lambda_l}^{\ \sigma_l}.$$

If there is an energy-momentum tensor $\Theta_{\mu\nu}$ satisfying the traceless condition, the generators of conformal transformations can be expressed using the conformal Killing vectors as

$$Q_\zeta = \int d^{D-1}\mathbf{x}\, \zeta^\lambda\, \Theta_{\lambda 0},$$

where $d^{D-1}\mathbf{x}$ is the spatial volume element. Indeed, using the conformal Killing equation (2-2) and the conservation equation $\partial^\mu \Theta_{\mu\nu} = 0$, we can show that $\partial_\eta Q_\zeta = -(1/D) \times \int d^{D-1}\mathbf{x}\, \partial_\lambda \zeta^\lambda \Theta^\mu_{\ \mu}$, thus when the energy-momentum tensor is traceless, the time-dependence disappears and the generator is conserved. Assigning $\zeta^\lambda_{T,L,D,S}$ (2-4) to ζ^λ, we obtain the following concrete expressions:

$$P_\mu = \int d^{D-1}\mathbf{x}\, \Theta_{\mu 0}, \quad M_{\mu\nu} = \int d^{D-1}\mathbf{x} \left(x_\mu \Theta_{\nu 0} - x_\nu \Theta_{\mu 0} \right),$$
$$D = \int d^{D-1}\mathbf{x}\, x^\lambda \Theta_{\lambda 0}, \quad K_\mu = \int d^{D-1}\mathbf{x} \left(x^2 \Theta_{\mu 0} - 2 x_\mu x^\lambda \Theta_{\lambda 0} \right). \quad (2\text{-}11)$$

As a simple example, calculations of the conformal algebra and the conformal transformation law in the case of a quantum free scalar field are given in the third section of Appendix B.

Finally, we give a differential equation that correlation functions satisfy. Conformal field theory is a theory with a conformally invariant vacuum $|0\rangle$, and such a vacuum is defined as a state that satisfies

$$Q_\zeta |0\rangle = 0, \qquad \langle 0| Q_\zeta = 0$$

for all generators $Q_\zeta\,(=Q_\zeta^\dagger)$. If any n conformal fields are simply expressed as O_{j_i} $(i=1,\dots,n)$, correlation functions of these fields satisfy $\langle 0|\,[Q_\zeta, O_{j_1}(x_1)\cdots O_{j_n}(x_n)]\,|0\rangle=0$. Thus,

$$\begin{aligned}&\delta_\zeta\langle 0|O_{j_1}(x_1)\cdots O_{j_n}(x_n)|0\rangle\\&=i\sum_{i=1}^{n}\langle 0|O_{j_1}(x_1)\cdots[Q_\zeta,O_{j_i}(x_i)]\cdots O_{j_n}(x_n)|0\rangle=0\end{aligned}$$

holds. This is an infinitesimal version of (2-9). For example, let O_{j_i} be a primary scalar field O_i with conformal dimension Δ_i and consider the case of D and K_μ as Q_ζ, we obtain

$$\sum_{i=1}^{n}\left(x_i^\mu\frac{\partial}{\partial x_i^\mu}+\Delta_i\right)\langle 0|O_1(x_1)\cdots O_n(x_n)|0\rangle=0,$$
$$\sum_{i=1}^{n}\left(x_i^2\frac{\partial}{\partial x_i^\mu}-2x_{i\mu}x_i^\nu\frac{\partial}{\partial x_i^\nu}-2\Delta_i x_{i\mu}\right)\langle 0|O_1(x_1)\cdots O_n(x_n)|0\rangle=0,$$

respectively, from the transformation law (2-10).

Correlation Functions and Positivity

Consider two-point correlation functions of traceless symmetric primary tensor fields of integer spin l defined by

$$W_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}(x-y)=\langle 0|O_{\mu_1\cdots\mu_l}(x)O_{\nu_1\cdots\nu_l}(y)|0\rangle. \qquad (2\text{-}12)$$

Letting Δ be conformal dimension of the field, it is generally expressed as

$$W_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}(x)=CP_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}(x)\left.\frac{1}{(x^2)^\Delta}\right|_{x^0\to x^0-i\epsilon},$$

where C is a constant and ϵ is an infinitesimal ultraviolet cutoff. The function $P_{\mu_1\cdots\mu_l,\nu_1\cdots\nu_l}$ is determined from the primary field condition.

In order to determine the form of the two-point correlation function, we use the conformal inversion (2-5), which is expressed as

$$x'_\mu=(Rx)_\mu=\frac{x_\mu}{x^2}.$$

This transformation gives $\Omega(x)=1/x^2$. Since it returns to its original form when it is operated twice, namely $R^2=I$, the inverse is given by $x_\mu=(Rx')_\mu$.

Primary scalar fields are transformed under the conformal inversion as

$$O'(x') = \Omega^{-\Delta}(x)O(x) = \left(x^2\right)^{\Delta} O(x).$$

It can be also written as $O'(x) = (x^2)^{-\Delta}O(Rx)$ by returning the argument to x. Here, we proceed with the discussion with the argument of O' as x'. Using this transformation law, the conformal invariance condition (2-9) expressed as $\langle 0|O'(x')O'(y')|0\rangle = \langle 0|O(x')O(y')|0\rangle$ yields

$$(x^2y^2)^{\Delta}\langle 0|O(x)O(y)|0\rangle = \langle 0|O(Rx)O(Ry)|0\rangle.$$

Noting that

$$\frac{1}{(Rx-Ry)^2} = \frac{x^2y^2}{(x-y)^2}, \tag{2-13}$$

we find that the two-point function of the primary scalar field is given by $1/(x-y)^{2\Delta}$ up to an overall coefficient. Restoring the ultraviolet cutoff, we get

$$\langle 0|O(x)O(0)|0\rangle = C\frac{1}{(x^2)^{\Delta}}\bigg|_{x^0\to x^0-i\epsilon} = C\frac{1}{(x^2+2i\epsilon x^0)^{\Delta}},$$

where $x^0 \neq 0$ and ϵ^2 is ignored. In the same way, we can determine the form of the three- and four-point functions of the primary scalar field (see the fourth section in Chapter 3).

Primary vector fields are transformed under conformal inversion as

$$O'_\mu(x') = \Omega(x)^{1-\Delta}\frac{\partial x^\nu}{\partial x'^\mu}O_\nu(x) = \left(x^2\right)^{\Delta} I_\mu{}^\nu(x)O_\nu(x),$$

where we introduce a function $I_{\mu\nu}$ of the coordinates x^μ defined by

$$I_{\mu\nu}(x) = \eta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2},$$

which satisfies $I_\mu{}^\lambda(x)I_{\lambda\nu}(x) = \eta_{\mu\nu}$ and $I^\mu{}_\mu(x) = D-2$. Therefore, the conformal invariance condition $\langle 0|O'_\mu(x')O'_\nu(y')|0\rangle = \langle 0|O_\mu(x')O_\nu(y')|0\rangle$ is expressed as

$$(x^2y^2)^{\Delta}I_\mu{}^\lambda(x)I_\nu{}^\sigma(y)\langle 0|O_\lambda(x)O_\sigma(y)|0\rangle = \langle 0|O_\mu(Rx)O_\nu(Ry)|0\rangle.$$

Noting that

$$\begin{aligned} I_\mu{}^\lambda(x)I_\nu{}^\sigma(y)I_{\lambda\sigma}(x-y) &= I_{\mu\nu}(x-y) + 2\frac{x^2-y^2}{(x-y)^2}\left(\frac{x_\mu x_\nu}{x^2} - \frac{y_\mu y_\nu}{y^2}\right) \\ &= I_{\mu\nu}(Rx-Ry), \end{aligned}$$