Controllability of Dynamic Systems

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The Green's Function Approach

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A. A.: To Academician of NAS of Armenia Sergei A. Ambartsumyan, on the occasion of his 95th birthday

As. Kh.: To my dearest mom

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Preface

Controllability, i.e., the ability of a system to be transmitted from a given initial state to a required terminal state by an admissible control within a finite time, is one of the most crucial characteristics of control systems. Controllability is of two main types: exact and approximate. The choice depends on how precisely the required terminal state is implemented. The existence of powerful computers and efficient numerical tools for linear and nonlinear equations allows the exact or approximate controllability analysis to be carried out numerically even for extremely complicated systems. Such analysis can bring with it a burdensome computational cost. However, in some special cases a simplifying step can considerably speed up the analysis. Such a step, for example, could be the determination of explicit dependence between state and control functions, i.e., a solution of the state constraints. While this is to some extent possible for linear systems, for nonlinear systems it is much more complicated.

The main motivation behind presenting our work to a wider scientific community is to illustrate how efficiently the Green's function method can be applied in controllability analysis of both linear and nonlinear dynamic systems and possibly initiate new studies in this direction. We illustrate this idea by a basic analysis of a few typical examples. Even though the examples were picked intuitively rather than systematically, they include some of the common issues which usually make the controllability analysis complicated: coordinate dependent material characteristics, unbounded domains, uncertainty in internal or external parameters, higher dimensions, specific non-linearities, etc. We also address the problem of determination of resolving control functions in an explicit form, which speeds up the controllability analysis further.

Generally speaking, due to, for example, modeling inaccuracies, random issues, uncertainties, etc., even by means of highly precise production technologies it is practically impossible to implement the desired state exactly. The terminal state implemented by the "best" choice of admissible control is more often "sufficiently" close to the desired state, rather than coinciding with it exactly. That is the motivation behind paying the most attention specifically to the *approximate* controllability. Nevertheless, possibilities of exact implementation of required states are shown as well.

In order to make the reading more engaging, the book is completely free of rigorous mathematical statements such as lemmas, propositions, theorems, as well as redundant long proofs, which can be found in the cited references. The focus is concentrated on the *ways* the developed approach can be applied in dealing with particular control problems. Thus, the book is mostly intended for researchers who are focused on the applications of control, as well as for engineers attempting to apply control techniques in their practice. It can also be useful for postgraduate students in mechanics, physics, engineering, applied mathematics, etc.

Even though we study the controllability of linear and nonlinear differential equations arising in many applied areas of science, including physics, production, mechanical, aerospace, civil and chemical engineering, hydrodynamics, information processing and transfer, communications, etc., we refrain from giving specific recommendations regarding particular real-life objects, processes or phenomena. The reason is very simple: we stand on mathematical ground and look at all those systems from a mathematical viewpoint. Nevertheless, we hope that the book will capture the interest of applied scientists towards the approach which will lead to its real-life implementations.

Ara S. Avetisyan
Asatur Zh. Khurshudyan
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List of Symbols

All notations and symbols are consistent over the book. Therefore, in order to assist readers who do not engage the book linearly, the main notions, functions, and symbols occurring in the book are defined separately here.

Symbol	Name & Definition
$\mathbb{R}$	Real numbers
$\mathbb{R}^+$	Positive real numbers
$\mathbb{N}$	Natural numbers
m, n, k...	$m, n, k \in \mathbb{N}$
Ω	An open subset of $\mathbb{R}^n$
$\mathring{\Omega}$	Interior of Ω
$\partial\Omega$	Boundary of Ω $\partial\Omega = \Omega \setminus \mathring{\Omega}$
$\overline{\Omega}$	Closure of Ω $\overline{\Omega} = \Omega \cup \partial\Omega$
$\operatorname{supp}(\cdot)$	Support of a function $\operatorname{supp}(f) = \overline{\{\boldsymbol{x}, f(\boldsymbol{x}) \neq 0\}}$

Symbol	Name & Definition
δ	Dirac delta function $\delta(x) = \begin{cases} 0, & x \neq 0, \\ \infty, & x = 0, \end{cases} \qquad \int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1,$ $\int_{-\infty}^{\infty} f(x)\,\delta(x-\xi)\,\mathrm{d}x = f(\xi)\ \forall\ \xi \in \mathbb{R}$
$\boldsymbol{\delta}$	Multidimensional Dirac function $\boldsymbol{\delta}(\boldsymbol{x}) = \prod_{k=1}^{n} \delta(x_k),\ \boldsymbol{x} = (x_1, \ldots, x_n)$
δ_m^n	Kronecker symbol $\delta_m^n = \begin{cases} 1, & m = n, \\ 0, & m \neq n \end{cases}$
θ	Heaviside function $\theta(x) = \begin{cases} 1, & x > 0, \\ \frac{1}{2}, & x = 0, \\ 0, & x < 0, \end{cases}$ $\theta(x) = \int_{-\infty}^{x} \delta(\xi)\,\mathrm{d}\xi$
sign	Sign function $\operatorname{sign}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0, \end{cases}$ $\operatorname{sign}(x) = 2\theta(x) - 1$

Symbol	Name & Definition
χ_Ω	Indicator function $\chi_\Omega(\boldsymbol{x}) = \begin{cases} 1, & \boldsymbol{x} \in \Omega, \\ 0, & \text{else}, \end{cases}$ $\chi_{[a,b]}(x) = \theta(x-a) - \theta(x-b)$
rect	Rectangular function $\operatorname{rect}(x) = \begin{cases} 0, & \lvert x\rvert > \frac{1}{2}, \\ \frac{1}{2}, & \lvert x\rvert = \frac{1}{2}, \\ 1, & \lvert x\rvert < \frac{1}{2}, \end{cases}$ $\operatorname{rect}(x) = \theta\left(\frac{1}{4} - x^2\right) = \theta\left(x + \frac{1}{2}\right) - \theta\left(x - \frac{1}{2}\right)$
erf	Gauss error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left[-\xi^2\right] \mathrm{d}\xi$
$\boldsymbol{x}$	State Variable $\boldsymbol{x} \in \Omega$
t	Time $t \in \mathbb{R}^+$
$\boldsymbol{w}$	State function $\boldsymbol{w} : \Omega \times \mathbb{R}^+ \to \mathbb{R}^m$
$\mathbf{W}$	State space Appropriate Hilbert space
$\boldsymbol{w}_0$	Initial state $\boldsymbol{w}_0 : \Omega \to \mathbb{R}^m$

Symbol	Name & Definition
$\boldsymbol{w}_T$	Required terminal state $\boldsymbol{w}_T : \Omega \to \mathbb{R}^m$
$\overline{\mathbf{W}}$	Restriction of $\mathbf{W}$ for fixed t Corresponding Hilbert space: $\boldsymbol{w}_0,\ \boldsymbol{w}_T \in \overline{\mathbf{W}}$
$\|\|\cdot\|\|_{\overline{\mathbf{W}}}$	Norm in $\overline{\mathbf{W}}$
$\boldsymbol{u}$	Distributed control $\boldsymbol{u} : \mathbb{R}^+ \to \mathbb{R}^k$
$\mathbf{U}$	Distributed control space Appropriate Hilbert space
$\boldsymbol{u}_b$	Boundary control $\boldsymbol{u}_b : \mathbb{R}^+ \to \mathbb{R}^m$
$\mathbf{U}_b$	Boundary control space Appropriate Hilbert space
$\mathcal{U}$	Set of admissible controls $\mathcal{U} = \{\boldsymbol{u} \in \mathbf{U},\ \|\boldsymbol{u}\| \leq \epsilon,\ \text{o.p.c.}\}$
$\mathcal{U}_{res}$	Set of resolving controls
$\boldsymbol{f}$	External influence $\boldsymbol{f} : \mathcal{U} \times \Omega \times \mathbb{R}^+ \to \mathbb{R}^m$
$\mathbf{F}$	Influence space Appropriate Hilbert space
L^1	L^1 space $L^1(\Omega) = \left\{\boldsymbol{f} : \Omega \to \mathbb{R}^m,\ \int_\Omega \|\boldsymbol{f}(\boldsymbol{x})\| \, \mathrm{d}\boldsymbol{x} < \infty\right\}$
$\|\|\cdot\|\|_{L^1}$	L^1-norm $\|\|\boldsymbol{f}\|\|_{L^1(\Omega)} = \int_\Omega \|\boldsymbol{f}(\boldsymbol{x})\| \, \mathrm{d}\boldsymbol{x}$

Symbol	**Name & Definition**
L^1_{loc}	L^1_{loc} space $L^1_{loc}(\Omega) = \left\{ \boldsymbol{f} : \Omega \rightarrow \mathbb{R}^m,\ \int_\Omega \lvert \boldsymbol{f}(\boldsymbol{x})\,\boldsymbol{\varphi}(\boldsymbol{x}) \rvert\, \mathrm{d}\boldsymbol{x} < \infty \right\}$ for each infinitely differentiable function $\boldsymbol{\varphi} : \Omega \rightarrow \mathbb{R}^m$ with $\operatorname{supp}(\varphi) \subseteq \Omega$
L^2	L^2 space $L^2(\Omega) = \left\{ \boldsymbol{f} : \Omega \rightarrow \mathbb{R}^m,\ \left[\int_\Omega \lvert \boldsymbol{f}(\boldsymbol{x}) \rvert^2 \mathrm{d}\boldsymbol{x} \right]^{\frac{1}{2}} < \infty \right\}$
$\lvert\lvert \cdot \rvert\rvert_{L^2}$	L^2-norm $\lvert\lvert \boldsymbol{f} \rvert\rvert_{L^2(\Omega)} = \left[\int_\Omega \lvert \boldsymbol{f}(\boldsymbol{x}) \rvert^2 \mathrm{d}\boldsymbol{x} \right]^{\frac{1}{2}}$
L^2_ν	Weighted L^2 space $L^2_\nu(\Omega) = \left\{ \boldsymbol{f} : \Omega \rightarrow \mathbb{R}^m,\ \left[\int_\Omega \lvert \boldsymbol{f}(\boldsymbol{x}) \rvert^2 \nu(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \right]^{\frac{1}{2}} < \infty \right\}$
$\lvert\lvert \cdot \rvert\rvert_{L^2_\nu}$	Weighted L^2-norm $\lvert\lvert \boldsymbol{f} \rvert\rvert_{L^2_\nu(\Omega)} = \left[\int_\Omega \lvert \boldsymbol{f}(\boldsymbol{x}) \rvert^2 \nu(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \right]^{\frac{1}{2}}$
L^∞	L^∞ space $L^\infty(\Omega) = \left\{ \boldsymbol{f} : \Omega \rightarrow \mathbb{R}^m,\ \sup_{\boldsymbol{x} \in \Omega} \lvert \boldsymbol{f}(\boldsymbol{x}) \rvert < \infty \right\}$
$\lvert\lvert \cdot \rvert\rvert_{L^\infty}$	L^∞ or sup-norm $\lvert\lvert \boldsymbol{f} \rvert\rvert_{L^\infty(\Omega)} = \sup_{\boldsymbol{x} \in \Omega} \lvert \boldsymbol{f}(\boldsymbol{x}) \rvert$
$\mathcal{D}$	State operator $\mathcal{D} : \mathbf{W} \rightarrow \mathbf{F}$

Symbol	Name & Definition
$\mathcal{B}$	Boundary operator $\mathcal{B} : \mathbf{W} \to \mathbf{U}_b$
$\mathcal{I}$	Operator of initial conditions $\mathcal{I} : \mathbf{W} \to \overline{\mathbf{W}}$
$\mathcal{T}$	Operator of terminal conditions $\mathcal{T} : \mathbf{W} \to \overline{\mathbf{W}}$
$\mathcal{R}_T$	Residue $\mathcal{R}_T(\boldsymbol{u}) = \|\|\mathcal{T}[\boldsymbol{w}] - \boldsymbol{w}_T\|\|_{\overline{\mathbf{W}}}$
Δ	Laplace operator $\Delta \boldsymbol{w} = \sum_{k=1}^{n} \frac{\partial^2 \boldsymbol{w}(\boldsymbol{x})}{\partial x_k^2}$
$\|\cdot\|$	Euclidean norm $\|\boldsymbol{x}\| = \sqrt{\sum_{k=1}^{n} x_k^2}$ for $\boldsymbol{x} \in \mathbb{R}^n$

1

Introduction

"Everything must be made as simple as possible. But not simpler."

– Albert Einstein

This introductory chapter begins by laying out the main concepts and problems the book is dealing with (Section 1.1), and then proceeds to the mathematical foundations of the solution technique including the description of the Green's function method for both linear and nonlinear systems (Sections 1.2 and 1.3). The technique itself is then described in detail in Section 1.4. The goal for the chapter is to set out the issues in an understandable manner for researchers without special mathematical training (especially for engineers and applied scientists) so it is bereft of mathematical formalism.

All the notations that are not common or well-known are explained within the text directly after being used.

1.1 Controllability

The majority of dynamic systems, e.g., vehicles, aircrafts, robots, production equipment, financial and biological processes, etc., are somehow controlled by a predetermined program or influence often referred to as *control programs* or simply *controls*. Controls are designed and implemented by special *controllers* attached to the system. Controllers have limited capabilities, so that restrained types of controls can be elaborated. Controls that a controller is able to use in design are often referred to as *admissible controls*.

Dynamic systems may need to be controlled for a variety of different purposes. Put simply, the aim of a control is to provide a stable transition of a dynamic system from a given state to a required terminal one within a fixed amount of time. Additional requirements, such as constraints on the state at intermediate time instants, constraints on control, mixed constraints, etc., may be considered as well. For a given system, situations may occur such that, even for a fixed range of system parameters, prescribed initial and terminal states, and a fixed control time, it is impossible to develop an admissible control providing the desired state transition. That is why before exploiting control systems, an overall examination of the ability to accommodate the desired state in the required time by means of attached controllers is carried out. This property is called *controllability*.

There are two main types of controllability—exact and approximate. A system is called *exactly* controllable if by a specific choice of admissible controls it can be transitioned from a given state to a required state exactly in a finite amount of time. There is a huge body of references devoted to the development of methods of exact control-