Financial Mathematics

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By

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PREFACE

In the education of financiers and economists in all universities of the world, an important role belongs to mathematical disciplines. Among these disciplines, financial mathematics occupies a very serious place, because it is the base for disciplines such as corporate finance, financial management, investment, taxation, business valuation, ratings, etc.

This textbook is intended for both undergraduate and post-graduate students studying the course "Financial Mathematics".

It differs from other textbooks in a detailed and accessible presentation with the derivation and proof of all statements and theorems and a much broader consideration of the issues raised.

In each chapter of the textbook, detailed practical examples are given, and at the end of each chapter, questions and tasks are given to control the degree of assimilation of the material and consolidation of what has been studied.

ABSTRACT

This textbook contains information on financial mathematics, knowledge of which is necessary not only for every financier, but also for any competent economist of a wide profile (and especially for financial analysts). It consists of seven chapters: "Interest theory", "Financial flows and rents", "Profitability and risk of financial transactions", "Portfolio analysis", "Bonds", "Capital structure: Modigliani-Miller theory" and "Capital structure: Brusov-Filatova-Orekhova theory". Each chapter contains many detailed practical examples, and at the end of each chapter questions and tasks for revision are given.

For undergraduate and graduate students of all financial and economic fields and profiles, including "Finance and Credit", "Accounting and Auditing", "Taxes and Taxation", "World Economy", etc. It will be useful for specialists of all financial and economic specialties (and especially for financial analysts) and for everyone who wants to master quantitative methods in finance and economics.

INTRODUCTION

In the education of financiers and economists in all universities of the world, an important role belongs to mathematical disciplines. Among these disciplines, financial mathematics occupies a very serious place, because it is the base for disciplines such as corporate finance, financial management, investment, taxation, business valuation, ratings, etc.

This textbook is intended for both undergraduate and post-graduate students studying the course "Financial Mathematics". It differs from other textbooks in its detailed and accessible presentation with derivation and proofs of all statements and a much broader consideration of the issues raised. So, for example, if in all standard textbooks the "Rule of 70" (the term for doubling the deposit at a given interest rate) is given only for the case of compound interest, then this textbook considers an analogue of "the rule 70" for the case of simple interest, the so-called "rule 100" and moreover the cases of increasing the deposit an arbitrary number of times for all types of interest rates: complex, simple, continuous and with multiple accruals of interest.

In the chapter "Portfolio Analysis", much attention is paid to the portfolio of two securities, the theory of which was developed by the authors specifically for this textbook. Mastering this simple Portfolio theory prepares students to study the more general portfolio theories of Markowitz and Tobin. Such a detailed and consistent presentation is aimed at the student's conscious, creative assimilation of the course program so that they have the opportunity to independently solve a wide variety of tasks and problems arising in practice.

In each chapter of the textbook, detailed practical examples are given, and at the end of each chapter, questions and tasks are given to control the degree of assimilation of the material and consolidation of what has been studied.

The textbook is written using a competence-based approach based on the lectures given by the authors for more than 15 years at the Financial University under the Government of the Russian Federation (Moscow).

The authors are enthusiasts of the introduction of mathematical methods in economics and finance. The understanding that finance is essentially a quantitative science, and quantitative methods play a crucial role in the training of financiers and economists of all profiles, is increasingly spreading among the specialists responsible for their training. An example of this understanding is the introduction at the Financial University under the Government of the Russian Federation on the initiative of the authors of the course "Financial Mathematics" as a mandatory bachelor's degree for students of all directions and profiles, which was an important step towards the integration of national financial and economic education into the global one, where the mathematical component of financial disciplines reaches 70% or more. Such extensive teaching of financial mathematics has led to the use of the knowledge gained during the study in the development of many special disciplines and ultimately to an improvement in the quality of education received by graduates.

The authors are planning to publish "Tasks on Financial Mathematics", the use of which together with this textbook will allow the reader to creatively and firmly master this course.

The authors also plan to publish the second part of the textbook, intended for master's students and including not general, but special questions for each specific master's program. It sets out issues such as the cost and structure of capital, the company's dividend policy, leasing and others for the "Financial Management" program, issues such as repayment of long-term loans, VaR and its application in banking and others for the "Banking industry" program, investments, modern models for evaluating the effectiveness of investment projects, financial markets and derivative financial instruments for the "Financial Markets" program, etc.

CHAPTER 1

THE THEORY OF INTEREST

Interest can be defined as compensation paid by the borrower to the lender for the use of capital. Therefore, interest can be considered as a rent that the borrower pays to the lender to compensate for losses from the latter's non–use of capital during the loan. In general, capital and interest do not necessarily represent the same commodity. However, we will consider capital and interest expressed in the same terms — in terms of money.

So, the lender provides the borrower with a certain amount of money; after the deadline, the borrower must repay the accrued amount equal to the amount of debt plus interest.

Effective interest rate is the amount paid to the borrower (investor) at the end of the accrual period for each unit amount borrowed (invested) at the beginning of the period.

Denoting the increased value of the unit amount at time t through a_t, the interest rate through i, and the increased value of the full amount through S_t, we have for the first accrual period

$$i_1 = \frac{(1+i)}{1} = \frac{a_1 - a_0}{a_0} = \frac{S_1 - S_0}{S_0}, \qquad (1.1)$$

for the n–th accrual period

$$i_n = \frac{a_n - a_{n-1}}{a_{n-1}} = \frac{S_n - S_{n-1}}{S_{n-1}}. \qquad (1.2)$$

From this formula it can be seen that the effective interest rate can change (and is changing) depending on the number of the accrual period, but, as will be shown below, in the very important and widely used case of compound interest, the effective interest rate for all accrual periods remains constant, i.e. for all $n \geq 1$.

1.1. Simple interest

Let S_0 be the initial amount of debt, i be the interest rate. In the simple interest scheme, S_0 will increase by iS_0 by the end of a single accrual period (usually a year), and the accrued amount of S_1 will be equal to

$$S_1 = S_0 + iS_0 = S_0(1+i). \quad (1.3)$$

By the end of the second accrual period, the initial amount of debt S_0 will increase by another iS_0 and the accrued amount will become

$$S_2 = S_1 + iP = S_0(1+2i). \quad (1.4)$$

By the end of the n–th accrual interval, the accrued amount will be

$$S_n = S_0(1+ni). \quad (1.5)$$

This formula is called **the simple interest formula**. The multiplier $(1 + ni)$ is called **the accrual coefficient (multiplier)**, and the value of ni is the interest rate for time n.

Thus, the sequence of incremented sums $S_1, S_2, \dots, S_n$ is an arithmetic progression with the initial term S_0 and the difference iS_0.

The interest for n years can be represented as

$$I_n = S_0 in. \quad (1.6)$$

Effective interest rate in the simple interest scheme

$$i_n = \frac{a_n - a_{n-1}}{a_{n-1}} = \frac{S_n - S_{n-1}}{S_{n-1}} = \frac{(1+in)-(1+i(n-1))}{1+i(n-1)} = \frac{i}{1+i(n-1)} \quad (1.7)$$

decreases with the growth of n.

If different interest rates $i_1, i_2, \dots, i_m$ are set at different intervals of interest accrual $n_1, n_2, \dots, n_m$, then the accrued amount S_n for the time $n_1 + n_2 + \dots + n_m$ will be equal to

$$S_n = S_0\left(1 + \sum_{k=1}^{m} n_k i_k\right). \quad (1.8)$$

The time of repayment of the loan may not be specified exactly, but may be a variable (for example, in the case of a cumulative deposit on demand). Then the formula of simple interest takes the following form:

$$S_t = S_0(1 + i(t - t_0)), \tag{1.9}$$

where t_0 — the moment when the loan was issued;

t — the moment of repayment of the debt with interest.

According to the formula (1.9), the accumulated sum is a linear function of time. The graph of this function on the "time—money" coordinate plane is a ray with a starting point (t_0, S_0) and an angular coefficient $S_0 i$. Obviously,

$$S_t{}' = S_0 i. \tag{1.10}$$

1.2. Compound interest

With the accrual of compound interest, **reinvestment, or capitalization of the interest received** occurs; thus, at the rate i, each subsequent accrued amount increases by a part i of the previous amount, which takes into account the interest accrued in previous periods.

In the S_0 compound interest scheme, by the end of a single interval, accruals will increase by iS_0, and the accrued amount of S_1 will be equal to

$$S_1 = S_0 + iS_0 = S_0(1 + i). \tag{1.11}$$

By the end of the second period, S_1 accruals will increase by iS_1 and the accrued amount will become

$$S_2 = S_1 + iS_1 = S_1(1 + i) = S_0(1 + i)^2. \tag{1.12}$$

By the end of the n–th accrual interval, the accrued amount will be

$$S_n = S_0(1 + i)^n. \tag{1.13}$$

The formula (1.13) is called **the compound interest formula**. Thus, the sequence of incremented sums $S_1, S_2, \ldots, S_n$ is a geometric progression with the initial term S_0 and the denominator of the progression $q = (1 + i)$.

Effective interest rate in the compound interest scheme for the n–th accrual period

$$i_n = \frac{a_n - a_{n-1}}{a_{n-1}} = \frac{S_n - S_{n-1}}{S_{n-1}} = \frac{(1+i)^n - (1+i)^{n-1}}{(1+i)^{n-1}} = i. \quad (1.14)$$

does not depend on n and is equal to the nominal one.

The increased sum S_n is proportional to the initial sum S_0. The proportionality coefficient $(1 + i)^n$is called **the multiplicative factor**.

Note that any moment of time t_k can be taken as a “zero” one. In this case, the formula (1.13) takes the form:

$$S_n = S_k (1 + i_T)^{n-k}. \quad (1.15)$$

где i_T — the interest rate for period T, constant for all periods.

Assuming $t = nT$, formula (1.15) can be rewritten as follows:

$$S(t) = S_0 (1 + i_T)^{t/T}. \quad (1.16)$$

Using the formula (1.16), it is possible to calculate the accrued amount at any time t (not necessarily a multiple of the accrual period T. In this case, it is said that **a continuous model** of a cumulative account is used in the compound interest scheme. In the future, unless otherwise specified, this particular model will be used.

The analogue of formula (1.15) in the continuous model is the following formula:

$$S(t) = S(\tau)(1 + i_T)^{(t-\tau)/T}. \quad (1.17)$$

When interest is accrued once a year (or more generally, if the interest accrual period coincides with the main time unit), formulas (1.16) and

(1.17) are simplified:

$$S(t) = S_0(1+i)^t; \tag{1.18}$$

$$S(t) = S(\tau)(1+i)^{t-\tau}, \tag{1.19}$$

where i — annual interest rate.

The interest for n years can be represented as

$$I_n = S_0[(1+i)^n - 1]. \tag{1.20}$$

1.3. Multiple interest accrual

If compound interest accrual occurs several times a year (m) (quarterly, monthly, etc.), then after t years the accrued amount will become equal to:

a) in the case of simple interest:

$$S(t,m) = S_0\left(1 + \frac{i}{m}mt\right) = S_0(1+it),$$

that is, the accrued amount does not depend on the multiplicity of accrual. This conclusion will be used by us when considering the continuous accrual of interest in the case of simple interest;

b) in the case of compound interest:

$$S(t,m) = S_0\left(1 + \frac{i}{m}\right)^{mt}. \tag{1.21}$$

In the next paragraph it will be shown that the effective interest rate in the compound interest scheme increases with increasing multiplicity of accrual and reaches a maximum with continuous accrual of interest. At the same time, the effective interest rate practically reaches saturation at $m \geq 6 \div 10$, i.e. above this multiplicity of accrual, the growth of the effective interest rate slows down sharply.

1.4. Continuous interest accrual

If the frequency of accrual of compound interest m increases indefinitely, then there is **a continuous accrual of interest**. In this case, after t years, the accumulated amount will be equal to:

a) in the case of simple interest:

$$S(t,\infty) = \lim_{m\to\infty} S(t,m) = \lim_{m\to\infty} S_0\left(1+\frac{i}{m}mt\right) = S_0(1+it),$$

that is, the accrued amount remains the same as with a single interest charge. This conclusion was made by us in the case of multiple accrual of interest. Both conclusions are related to the fact that with any multiplicity of interest accrual, accrual is made on the initial amount in proportion to the time of the deposit;

b) in the case of compound interest:

$$S(t,\infty) = \lim_{m\to\infty} S(t,m) = \lim_{m\to\infty} S_0\left(1+\frac{i}{m}\right)^{mt} =$$

$$= \lim_{m\to\infty} S_0\left(1+\frac{i}{m}\right)^{mit/i} = S_0e^{it}. \qquad (1.22)$$

The interest rate i in the formula (1.22) is also called **the intensity of the growth rate** and is usually denoted by the letter δ. With this in mind, this formula can be written as:

$$S(t) = S_0e^{\delta t}. \qquad (1.23)$$

The intensity of the growth rate δ is characterized by the relative increase in the accrued amount over an infinitesimal period of time

$$S(t) = S_0 \cdot e^{\delta t} \cdot \delta = S(t) \cdot \delta, \qquad (1.24)$$

or

$$\frac{dS(t)}{S(t)} = \delta \cdot dt. \qquad (1.25)$$

If the intensity of the growth rate depends on time, then S(t) can be obtained as a solution of the differential equation (1.25). Finding the integral of both parts (1.25), we get

$$\ln S(t) - \ln S_0 = \int_0^t \delta \, dt. \qquad (1.26)$$

It means that

$$S(t) = S_0 e^{\int_0^t \delta \, dt}. \qquad (1.26)$$

Example 1.1. The bank has a deposit of 1000 \$ at 10% per annum under the compound interest scheme. Find the amount of the deposit in three years when interest is accrued 1, 4, 6, 12 times a year and in the case of continuous interest accrual.

By the formula (1.21) we have

$$S_{3/1} = 1000(1 + 0.1)^3 = \$1{,}331,$$
$$S_{3/4} = 1000\left(1 + \frac{0.1}{4}\right)^{3\cdot 4} = \$1{,}344.9,$$
$$S_{3/6} = 1000\left(1 + \frac{0.1}{6}\right)^{3\cdot 6} = \$1{,}346.5,$$
$$S_{3/12} = 1000\left(1 + \frac{0.1}{12}\right)^{3\cdot 12} = \$1{,}348.2.$$

In the case of continuous accrual of interest, the formula (1.22) must be used

$$S_{3/\infty} = 1000e^{0.1\cdot 3} = \$1{,}349.6. \qquad (1.27)$$

Interest for three years amounted to (\$):

— with a single accrual of interest — 331;
— with a four–time accrual — 344.9;
— with a six–time accrual — 346.5;
— at twelve–time accrual —348.2;
— with continuous accrual— 349.6.

We come to the conclusion that the accrued amount, as well as the amount of interest money, in the compound interest scheme increases with increasing multiplicity of accrual and reaches a maximum with continuous accrual of interest. Moreover, the growth rate of both values decreases with an increase in the multiplicity of accrual. (For proof of these facts, see paragraph 1.13.)

Example 1.2. An amount of $3,000 was put on the bank deposit on March 10 at 15% per annum under the compound interest scheme. What amount will the depositor receive on October 22?

We use the formula (1.13) for the accrual according to the scheme of compound interest:

$$S_n = S_0(1+i)^n$$

Duration of the financial transaction (in fractions of the period)

$$n = \frac{t}{T} = \frac{20 + 30 \cdot 6 + 22}{365} = 0.608$$

(it is assumed that there are 30 days in a month, 365 in a year), so we have

$$S_n = S_0(1+i)^n = 3{,}000(1+0.15)^{0.608} = \$3{,}266.07.$$

So, on October 22, the depositor will receive $3,266.07.

1.5. Equivalence of interest rates in the compound interest scheme

Let's consider interest rates, using which a model of the percentage growth of the accrual in the compound interest scheme can be described.

If the accrual rate i for the accrual period T is specified, then

$$S_t = S_0(1+i)^{\frac{t}{T}}. \quad (1.28)$$

If the annual rate j and the multiplicity of accrual (during the year) p are specified, then

$$S_t = S_0(1 + j/p)^{pt}. \quad (1.29)$$

In this case, it is said that j is **the nominal rate**.

With continuous accrual of interest

$$S_t = S_0 e^{\delta t}. \quad (1.30)$$

And the intensity of the growth rate δ is also called **the continuous nominal rate**.

Finally, if **the effective rate** i_{eff} is specified, the accrued amount is determined by the formula

$$S_t = S_0\left(1 + i_{eff}\right)^t. \quad (1.31)$$

Formulas (1.28)—(1.31) have the form:

$$S_t = S_0 a^t. \quad (1.32)$$

where a —the corresponding (normalized) **accrual coefficient**.

In each case, a is obtained as an annual accrual factor.

Rates are called **equivalent** if they have the same growth coefficients. This means that with the same initial amount, the amounts accumulated by any point in time t at equivalent rates are the same.

The growth coefficient a and the effective i_{eff} rate are related by a simple ratio

$$a = 1 + i_{eff} \quad (1.33)$$

With this in mind, we can say that the rates are equivalent if the effective rates equivalent to them coincide.

It is not difficult to specify the ratios that ensure the equivalence of rates of various types.

If j is the annual rate at the multiplicity of accrual p, then it is equivalent to the rate $iT = i_{1/p} = j/p$ for the period $T = 1/p$. The equivalent effective rate is determined by the formula

$$i_{eff} = \left(1 + \frac{j}{p}\right)^p - 1, \quad (1.34)$$

or

$$i_{eff} = (1 + i_T)^{1/T} - 1. \quad (1.35)$$

Accordingly,

$$j = p\left(\left(1 + i_{eff}\right)^{\frac{1}{p}} - 1\right); \quad (1.36)$$

$$i_T = \left(1 + i_{eff}\right)^T - 1. \quad (1.37)$$

With continuous accrual of interest, we get:

$$i_{eff} = e^{\delta} - 1, \quad (1.38)$$

$$\delta = \ln\left(1 + i_{eff}\right). \quad (1.39)$$

iT and *iT* interest rates with accrual periods T_1 and T_2, respectively, are equivalent if

$$\left(1 + i_{T_1}\right)^{\frac{1}{T_1}} = \left(1 + i_{T_2}\right)^{\frac{1}{T_2}}. \quad (1.40)$$

If different interest rates $i_1, i_2, \ldots, i_m$ are set at different intervals of interest accrual $n_1, n_2, \ldots, n_m$, then the accumulated amount Sn for the time $n_1 + n_2 + \ldots + n_m$ will be equal to

$$S_n = S_0(1 + i_1)^{n_1}(1 + i_2)^{n_2} \ldots (1 + i_m)^{n_m} = S_0 \prod_{k=1}^{m}(1 + i_k)^{n_k}. \quad (1.41)$$

1.6. Comparison of accruals at simple and compound interest rates

At the same interest rate, the increase according to the simple interest scheme is more advantageous for an accrual period of less than a year. For an accrual period of more than a year, it is more advantageous to be accrued according to the compound interest scheme (Fig. 1.1). For proof, it is sufficient to show that

$$f(t) = (1+i)^t < g(t) = 1 + ti, \quad if\ 0 < t < 1;$$

$$f(t) = (1+i)^t > g(t) = 1 + ti, \quad if\ t > 1$$

For the second order derivative of the function $f(t)$we have $f''(t) = =\ln^2(1+i)(1+i)^t > 0$, therefore, $f(t)$ is a convex down function at $t > 0$, and $g(t) = 1 + it$ is a chord to $f(t)$, since the equation $f(t) = g(t)$ or $(1+i)^t = 1 + + ti$ has two solutions: $t = 0$ and $t = 1$. Hence $(1 + i)^t < 1 + ti$ if $0 < t < 1$, and $(1 + i)^t > 1 + ti$ if $t > 1$.

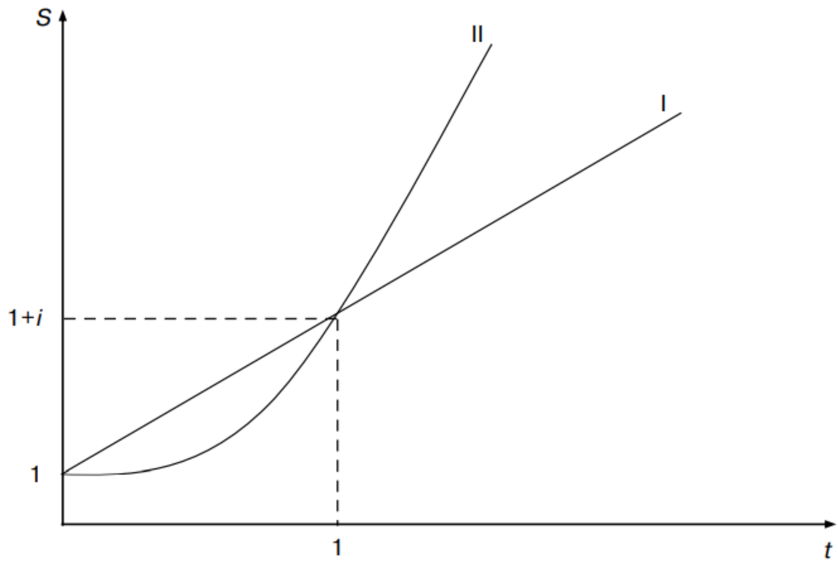

Figure 1.1. Accrual at simple (I) and complex (II) interest rates

Important notice 1

When interest is calculated once a year, simple interest is more effective than compound interest with a deposit term of up to one year, and compound interest is more effective with a deposit term of more than one year. This can be seen from Fig.1.1.

Does this condition change with multiple accrual percent and if it changes, then how. We leave readers to verify the following: with multiple accrual interest, simple interest is more effective than compound interest before the first accrual of interest. By other words at monthly accrual of interest, simple interest is more effective than compound interest during the first month; at quarterly accrual of interest, simple interest is more effective than compound interest during the first quarter; when interest is accrual semi–annually, simple interest is more effective than compound interest during the first half of the year, and so on.

Important notice 2

Concerning the continuous interest, they are more effective than simple or compound interest for any term of the deposit.

1.7. Discounting and interest deduction

Discounting and interest deduction are in a certain sense the reverse of interest accrual. There are *mathematical discounting* and *bank accounting*.

Mathematical discounting allows you to find out what initial amount S_0 needs to be invested in order to receive, after t years, the amount $S_t$when interest is accrued on S_0 at the rate i.

In the case of simple interest

$$S_o = S_t/(1 + ti). \tag{1.42}$$

In the case of compound interest

$$S_o = S_t/(1 + i)^t. \tag{1.43}$$

In the case of continuous accrual of interest

$$S_o = S_t / e^{\delta t}. \quad (1.43)$$

The value S_0 is called **the present value** of the value S_t. The values i and δ, which were previously called interest rates, now mean **discount rates**.

Bank accounting is the purchase by a bank of monetary obligations at a price less than the nominal amount specified in them.

An example of monetary obligations is **a promissory note** — a promissory note containing an obligation to pay a certain amount of money (nominal value) within a certain period.

In the case of a bank purchase of a bill, they say that the latter is *taken into account*, and the amount is paid to the client

$$S_n = S_o - I_n \quad (1.45)$$

where S_0 — nominal amount of the promissory note;

S_n — the purchase price of the promissory note by the bank for n years before maturity;

I_n — discount, or the bank's income (interest money).

$$I_1 = S_o d. \quad (1.46)$$

where d — discount rate (as a rule, through d we will further denote the discount rate).

The discount rate can be simple and complex, depending on which scheme is used — simple or compound interest. In the case of simple interest, the sequence of amounts remaining after the discount $\{S_n\}$ forms a decreasing arithmetic progression with a common term $S_n = S_0(1 - nd)$ equal to the amount that the client will receive n years before repayment.

In the case of compound interest, the sequence of amounts remaining after the discount $\{S_n\}$ forms a decreasing geometric progression with a common term $S_n = S_0(1 - d)^n$ equal to the amount that the client will receive *n* years before repayment.

1.7.1. Comparison of discounting at complex and simple discounting rates

For the bank, the discounting situation is the inverse of the accrual. For example, if the accounting period is less than one year, it is more profitable for the bank to discount at a complex discount rate (Figure 1.2) (the accrual — at a simple one (Figure 1.1)), and if the accounting period is more than one year — at a simple discount rate (Figure 1.2) (the accrual — at a complex one (see Figure 1.1)).

For proof, it is sufficient to show that

$$f(t) = (1 + d)^t < g(t) = 1 - td, \quad if\ 0 < t < 1;$$

$$f(t) = (1 + d)^t > g(t) = 1 - td, \quad if\ t > 1$$

For the second derivative of the function *f(t)* we have f ''(t) = $\ln^2(1-d)\bullet \bullet(1 - d)^t > 0$, hence *f(t)* is a convex down function at $t > 0$, and $g(t) = 1-id$ is a chord to *f(t)*, since the equation $f(t) = g(t)$ or $(1 - d)^t = 1 - dt$ has two solutions: $t = 0$ and $t = 1$. Hence, $(1 - d)^t < 1 - dt$ if $0 < t < 1$, and $(1 - d)^t > 1 - dt$ if $t > 1$.

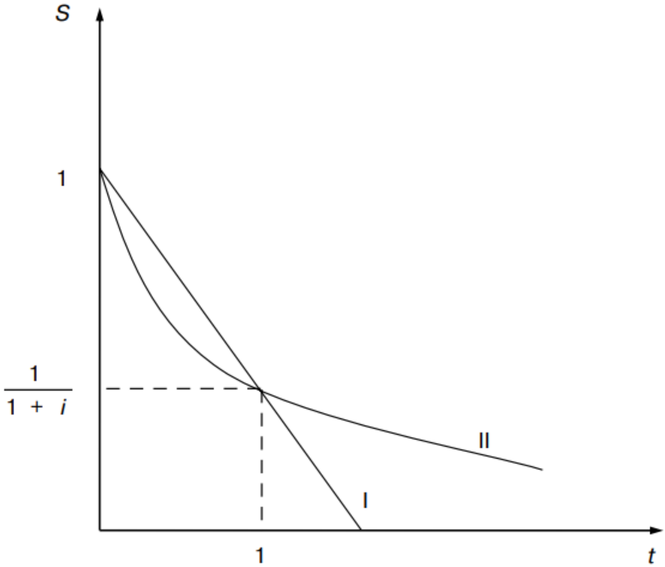

Figure 1.2. Discounting at simple (I) and complex (II) interest rates

1.7.2. Effective discount rate

Let d_{eff} be the annual (effective) discount rate (discount rate) with a multiplicity of accrual m. The equivalent effective discount rate is determined based on the equivalence principle

$$S_o\left(1-d_{eff}\right)^n = S_o\left(1-\frac{d}{m}\right)^{n\cdot m}, \tag{1.47}$$

hence

$$1-d_{eff} = \left(1-\frac{d}{m}\right)^m, \tag{1.48}$$

or

$$d_{eff} = 1-\left(1-\frac{d}{m}\right)^m, \tag{1.49}$$

Inversely, the discount rate d is expressed in terms of the effective discount rate d_{eff}:

$$d = m\left(1 - \sqrt[m]{1 - d_{eff}}\right). \quad (1.50)$$

The discount rate *d* and the interest rate *i* lead to the same result over a period of time *t* if

$$S_0(1 + it) = S_t \text{ and } S_0 = S_t\,(1 - dt), \quad (1.51)$$

$$(1 + it)(1 - dt) = 1. \quad (1.52)$$

The last equality can be transformed as follows:

$$d = \frac{i}{1+it}; \; i = \frac{d}{1-dt}. \quad (1.53)$$

We can also write down the relationship between the nominal rates of increment and discounting

$$\left(1 + \frac{i}{m}\right)^m = \left(1 - \frac{d}{p}\right)^{-p}, \quad (1.54)$$

since both parts of the equation are equal to $(1 + i)$.

If $m = p$, we have

$$\left(1 + \frac{i}{m}\right)^m = \left(1 - \frac{d}{m}\right)^{-m}, \quad (1.55)$$

hence

$$\frac{i}{m} - \frac{d}{m} = \frac{i}{m} \cdot \frac{d}{m}. \quad (1.56)$$

If different discount rates $i_1, i_2, \ldots, i_m$ are set at different discount intervals $n_1, n_2, \ldots, n_m$, then Sn for the time $n_1 + n_2 + \ldots + n_m$ will be equal to

$$S_n = S_0(1 + i_1)^{-n_1}(1 + i_2)^{-n_2} \ldots (1 + i_m)^{-n_m} =$$

$$= S_0 \prod_{k=1}^{m}(1 + i_k)^{-n_k}, \quad (1.57)$$