Kinematics and Dynamics of Galactic Stellar Populations

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Preface

Stellar dynamics is an interdisciplinary field where mathematics, physics, and astronomy overlap. It describes systems of stars considered as many point mass particles whose mutual gravitational interactions determine their orbits. Theses interactions may arise from the smoothed-out stellar distribution of matter, which are then given through a gravitational potential, and from the effect of the stellar encounters. The collisional relaxation time is used to measure how long will it take before the cumulative effect of stellar encounters prevents us from considering the stars as independent, conservative dynamical systems. In large stellar systems, like a galaxy, the relaxation time is long and they may be assumed to be in statistical equilibrium according to specific phase space density and potential functions. This approach is generally done from the *analytical dynamics* viewpoint. That is, the stellar system is described as a conservative dynamical system from the canonical equations through a Hamiltonian function, and the hydrodynamical flow in the phase space is obtained according to Liouville's theorem. In this monograph we shall focus our attention on analytical stellar dynamics, by considering the stellar system as a fluid. In contrast, in small globular clusters, and in processes of violent relaxation producing rapid fluctuations of the gravitational field, the collisions cannot be omitted, and *statistical mechanics* is generally used to describe the dynamics of the interacting particles through a many particle distribution function.

Analytical stellar dynamics has its origins in the early 20th century, when the kinetic theory of gases was adapted to astronomical problems by J.H. Jeans. He showed that, under some regularity conditions, the fundamental equation of stellar dynamics is equivalent to the collisionless Boltzmann equation, so that the Liouville theorem is satisfied. Then, velocity moments of the collisionless Boltzmann equation yield the stellar hydrodynamic equations. These equations, written in a comoving reference frame, are comparable to the equations of motion of a compressible viscous fluid.

However, because short-range atomic interactions dominate fluids, it is a much better approximation to truncate the moment equations at low order for fluids (i.e. continuity and Navier-Stokes equations) than it is for stellar systems. In addition, the general stellar hydrodynamical equations are anisotropic in their spatial and velocity coordinates. Therefore, higher-order hydrodynamic equations are non-negligible for stellar dynamics.

In the forties, S. Chandrasekhar gave an alternative formulation to explain the dynamics of collisionless systems and, in addition, he introduced a new statistical approach for collisional systems through a dynamical friction mechanism. Chandrasekhar's alternative approach for collisionless systems is known as Jeans' inverse problem and it is a functional approach for the phase space density function based in the assumption that the *residual velocity distribution* of any stellar population in statistical equilibrium satisfies a generalised ellipsoidal law.

During the second half of last century, J.J. de Orús produced a rigorous mathematical formulation of Chandrasekhar's theory which was collected in his *Notes on Galactic Dynamics* for the Astronomy Department in the University of Barcelona. He and his disciples thoroughly studied the direct and inverse Jeans' problems. It was proven that, if the Chandrasekhar equations are fulfilled, the continuity equation and the Navier-Stokes equation are also satisfied and, even more, that the Chandrasekhar equations could be derived from the first four hydrodynamic equations. Solutions to the Chandrasekhar equations were given under hypotheses of axial (rotational) and point-axial symmetry, and for stellar population mixtures.

The aim of the current monograph is to review, update, and make these topics available to a broader audience. It is a fascinating area that addresses issues on dynamical systems, information theory, numerical analysis, partial differential equations, probability, statistics, tensor algebra, and vector calculus, among other topics, in addition to astronomy and physics subjects.

These nine chapters altogether provide the reader with a quite complete review of what are the main problems in this area at a level of postgraduate course. The first two chapters are devoted to the Jeans' direct problem, where the full mathematical expression of an arbitrary n-order stellar hydrodynamic equation, either depending on the pressures or on the comoving moments, is derived. In this way, the stellar hydrodynamic equations can be compared to the equations of fluid dynamics, and general closure conditions can be studied in order to build up a dynamical model from a finite number of equations and variables, generally known as closure problem.

The third and fourth chapters deal with the Jeans' inverse problem in relation to the long-standing closure problem, which is one of the classic, unsolved problems in fluid dynamics, discovered even earlier, in the nineteenth century, in ordinary hydrodynamics by O. Reynolds. The equivalence of the Chandrasekhar equations and the stellar hydrodynamic equations is discussed by proving that, for a generalised ellipsoidal velocity distribution, some moment recurrence relationships act as closure conditions making the infinite hierarchy of the hydrodynamic equations equivalent to the collisionless Boltzmann equation. This result is generalised to maximum entropy velocity distributions and to any velocity distribution function depending on a polynomial function in the velocity variables.

Chapters five and six focus on the distribution function and the moments problem. The maximum entropy approach for the solution of inverse problems, first introduced by E.T. Jaynes, illustrates how the velocity distribution function is connected to the eventual asymmetries collected through its population moments. The density function maximising Shannon's information entropy provides the simplest and smoothest approach to the distribution function that fulfils a provided set of moment constraints, and gives a very good estimation of the density function and of its velocity derivatives involved in the collisionless Boltzmann equation. In particular, if an extended set of moments is available, the parameter estimation of the distribution function may be simply done by solving a linear system of equations. Several numerical applications of this functional approach, either to complete or truncated distributions, are presented to show how the above mathematical methods are able to describe the main kinematical features of the neighbourhood stars.

As an alternative approach, in the seventh chapter, instead of using a higher-degree polynomial along with a maximum entropy function, the mixture model of Schwarzschild (Gaussian) density functions is studied in connection with the moments problem. This approach is useful to describe large-scale kinematic structures of the Galactic disc associated with kinematic stellar populations, that has been a very active research field during the last decades.

Finally, in the last two chapters, Chandrasekhar's dynamical models for axisymmetric and point-axial symmetric systems are studied, with a particular application to the superposition problem, which is the appropriate approach to the actual case of a Galaxy composed of several stellar populations. Under a common potential, a finite mixture of ellipsoidal velocity distributions satisfying the collisionless Boltzmann equation provides a set of integrability conditions that may constrain the population kinematics. These conditions determine which potentials are connected with a more flexible superposition of stellar populations.

The author wishes to record his gratitude to the late Prof. Juan J. de Orús. His earliest Notes on Galactic Dynamics and his always stimulating comments were, some years ago, the origin of the current notes.

Chapter 1

Analytical dynamics

1.1 Introduction

The aim of the first two chapters is to provide the complete mathematical expression for an arbitrary *n*-order hydrodynamic equation depending on the pressures, or alternatively on the comoving moments, without any additional hypotheses.

The stellar hydrodynamic equations have been used in a number of works on galactic dynamics to study the stellar mass and velocity distributions, either from an analytical viewpoint (e.g., Vandervoort 1975, Hunter 1979, Evans & Lynden-Bell 1989, Evans et al. 2000, van de Ven et al. 2003, Evans et al. 2015, An & Evans 2016) or as a model for numerical simulations to investigate the shape of the velocity distribution, or to reproduce the spiral structure of galactic discs as an alternative way to the *N*-body approach (e.g., Korchagin et al. 2000, Orlova et al. 2002, Vorobyov & Theis 2006). However, only equations of mass, momentum and, in few cases, energy transfer are generally handled, and, in most cases, axial symmetry, steady-state stellar system, and other hypotheses are assumed. There are few works that, in a mathematical aspect, have gone beyond such a basic assumptions. Sala et al. (1985) proposed a general expression for the *n*-order equation, without steadiness and axisymmetry, although it was written depending on the absolute, non-comoving moments of the stellar velocity distribution, where, by substitution of the moments as a series of the pressures, they obtained a general but non-explicit expression of the equations. The explicit equations were, in the end, specifically written for orders $n = 0, 1, 2, 3$. However, it is well-known that stellar hydrodynamic equations are physically meaningful when they can be compared with the ordinary hydrodynamic equations of a compressible, viscous fluid, and this is only possible when they are written in terms of the tensors of comoving moments or of pressures, in the reference frame associated with the local centroid. Often, these expansions or computational procedures are provided instead of their explicit expression, and they are later used to simplify and to close the system of equations, for example to study a cool, pure rotating disc (Aoki 1985, Amendt & Cuddeford 1991).

On the other hand, the work by Cuddeford & Amendt (1991) had also a general and more interesting mathematical scope, although it was restricted to steady-state systems, amid other hypotheses. They studied higher-order stellar hydrodynamic equations, by using central velocity moments up to eighth-order, and they investigated some quite general conditions over the velocity distribution in order to close the infinite hierarchy of the moment equations.

The general expression for such an arbitrary order hydrodynamic equation in the comoving frame was first derived by Cubarsi (2007, 2013). It should be taken as a starting point in forthcoming works either to use improved observational data or to carry out more exhaustive numerical simulations. In addition, the exact n-order equation is also essential to study more general closure conditions or, under unrestrictive assumptions, for building up more accurate dynamical models from a finite number of equations and variables.

Actual kinematic data (ESA 1997, Nordtröm et al. 2004) do not support any more the hypotheses of axisymmetry, steadiness, or pure galactic rotation (Cubarsi & Alcobé 2006). In addition, newer missions such as the RAdial Velocity Experiment (RAVE) survey (Siebert et al. 2011, Zwitter et al. 2008, Steinmetz et al. 2006) represent a major improvement, since the three velocity components are available for the largest number of stars ever collected, where an unbiased radial velocity component will provide essential information to kinematic and dynamic studies of the Galaxy.

1.2 Jeans' problems

From a macroscopic approach, a stellar system is described by giving its distribution in the phase space, which consists in couples of three-dimensional vectors $\boldsymbol{r}$ and $\boldsymbol{V}$ representing star position and velocity, measured from an inertial reference system. The stellar distribution is then given through the phase space density function $f(t, \boldsymbol{r}, \boldsymbol{V})$, which is assumed as continuous and differentiable in nearly every point, providing, at time t, the number of stars with position within the range $\boldsymbol{r}$ and $\boldsymbol{r} + d\boldsymbol{r}$, and velocity between $\boldsymbol{V}$ and $\boldsymbol{V} + d\boldsymbol{V}$.

It is generally assumed that the Galaxy is at present in a state in which each star can be idealised as a conservative dynamical system to a very high degree of accuracy. In general, the forces acting in the system can be associated with a gravitational potential function per unit mass $\mathcal{U}(t, \boldsymbol{r})$, possibly non-stationary, so that the motion of a star is described in a Cartesian coordinates system by the Hamiltonian system of equations

$$\dot{\boldsymbol{r}} = \boldsymbol{V}, \quad \dot{\boldsymbol{V}} = -\mathcal{U}(t, \boldsymbol{r}). \tag{1.1}$$

For the whole stellar system, the collisionless Boltzmann equation is satisfied, so that the phase space density function $f(t, \boldsymbol{r}, \boldsymbol{V})$, with $(t, \boldsymbol{r}, \boldsymbol{V}) \in R \times \Gamma_r \times \Gamma_V$, by using the Stokes operator $\frac{D(\cdot)}{Dt}$[1], fulfils

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \boldsymbol{V} \cdot \nabla_{\boldsymbol{r}} f - \nabla_{\boldsymbol{r}} \mathcal{U} \cdot \nabla_{\boldsymbol{V}} f = 0. \tag{1.2}$$

The above equation is sometimes referred as Vlasov equation, Liouville equation, Boltzmann equation, or Jeans equation; however, Hénon (1982) clarifies the appropriate terminology.

The collisionless Boltzmann equation is a consequence of the Hamiltonian flow, which preserves volume, i.e. satisfies the Liouville theorem: the density of any element of phase space remains constant during its motion.

Jeans showed that the *fundamental equation of stellar dynamics* was a particular case of the Boltzmann equation from the kinetic theory of gases,

$$\frac{\partial f}{\partial t} + \dot{\boldsymbol{r}} \cdot \nabla_{\boldsymbol{r}} f + \dot{\boldsymbol{V}} \cdot \nabla_{\boldsymbol{V}} f = C(f, f), \tag{1.3}$$

where the collision term of the right-hand side may be assumed to be null in two cases. First, if the effect of the irregular forces, such as star encounters, is negligible. Second, if the phase density is invariant with respect to the irregular forces, that is, when the number of points leaving any space volume as a result of encounters is balanced by those which enter the volume for the same reason. In both cases, the Liouville theorem is satisfied.

Hilbert (1912) gave an equivalent mathematical condition to neglect the collision term $C(f, f)$, when it is orthogonal to 1, $\boldsymbol{V}$, and $|\boldsymbol{V}|$.

[1]The Stokes operator $\frac{D(\cdot)}{Dt}$ is generally used to simplify the notation of the Lagrangian derivative $\frac{\partial(\cdot)}{\partial t} + \boldsymbol{V} \cdot \nabla_{\boldsymbol{r}}(\cdot) + \dot{\boldsymbol{V}} \cdot \nabla_{\boldsymbol{V}}(\cdot)$.

The collisional relaxation time is long in large stellar systems. The time of relaxation for stellar encounters in the solar neighbourhood is greater than 10^{13} years (Binney & Tremaine 2008), while the galactic rotation period is about 10^8 years. Hence, the encounters are entirely unimportant. The collisions cannot be omitted in a globular cluster which contain 10^5 stars, but for a galaxy of 10^{11} stars, the relaxation time turns out to be much larger than the age of the universe, and the encounters can be neglected.

The collisionless Boltzmann equation may be regarded from two different viewpoints. It is either a linear and homogeneous partial differential equation for f, for a given potential $\mathcal{U}$, which is known as Jeans' direct problem, or a linear non-homogeneous partial differential equation for $\mathcal{U}$, where the density function f is already known, which is called Jeans' inverse problem. Both approaches have been largely studied since Eddington (1921) and Oort (1928), and among other works, those of Vandervoort (1979), de Zeeuw & Lynden-Bell (1985), Bienaymé (1999) and Famaey et al. (2002) may be pointed out.

Obviously, neither the phase space density function nor the potential are observable quantities, while we do have enough large data sets of the full space motions in the solar neighbourhood for different types of stars to compute the kinematic statistics of the distribution. Then, in order to isolate information about the spatial properties of the stellar system, the collisionless Boltzmann equation may be integrated over the velocity space, or in a more general way, it may be multiplied through by any powers of the velocities before integrating, and each choice of powers leads to a different equation which involves the kinematic statistics describing the stellar system for fixed time and position, which are the mean velocity and the moments of the velocity distribution. Such a strategy, which is usually referred as moment or fluid approach, provides us with an infinite hierarchy of stellar hydrodynamic equations, which could be used as a dynamical model to study the stellar system, on condition that some closure relationships were available in order to work with a finite number of equations and unknowns.

1.3 Isolating integrals

According to Jeans' direct problem, Eq. 1.2 is a linear and homogeneous partial differential equation for f, for a given potential $\mathcal{U}$, whose subsidiary

Lagrange system of equations is

$$\frac{dr_1}{V_1} = \frac{dr_2}{V_2} = \frac{dr_3}{V_3} = \frac{dV_1}{\frac{\partial \mathcal{U}}{\partial r_1}} = \frac{dV_2}{\frac{\partial \mathcal{U}}{\partial r_2}} = \frac{dV_3}{\frac{\partial \mathcal{U}}{\partial r_3}} = dt \tag{1.4}$$

An immediate consequence of the Liouville theorem is that if $I_1, I_2, ..., I_6$ are any six functional independent integrals of the stellar motion for a given potential satisfying the equations in Eq. 1.4, then the phase space density function must be of the form $f(t, \boldsymbol{r}, \boldsymbol{V}) = f(I_1, I_2, ..., I_6)$, where the quantity on the right-hand side stands for an arbitrary function of the specified arguments, on the condition that the mass of the system be finite and that the density in the phase space be non-negative. The phase space density function is itself an integral of motion. The integrals of motion univocally determine the orbit of any star in the phase space.

However, the phase density, by its physical significance, must be a one-valued function of the six phase coordinates. Therefore, only the integrals satisfying the condition of being one-valued in phase space, which are called *isolating integrals*, can appear as an argument of the phase density, although they may take several values in the space of integrals of motion. In 1953 G. Kuzmin was the first in suggesting this fact, and Lynden-Bell (1961) provided a rigorous demonstration that a continuous phase space density function must be independent of any non-isolating integral almost everywhere.

More precisely, if in a bounded region of the phase space the equation $I_k(\boldsymbol{r}, \boldsymbol{V}) = C_k$ can be solved with respect to every variable and gives a finite number of solutions, then the integral is called isolating (e.g., Contopoulus 1963). On the other hand, if there exist at least one accumulation point at a finite distance, the integral is called non-isolating. Isolating integrals are important because they constrain the shapes of orbits by one dimension in the phase space. Analytic integrals in a simply connected region including the phase space are isolating, but non-classic integrals, which are implicit in a numerical integration of an orbit, are usually non-isolating.

Thus, if the stellar system is time-independent, the phase space remains decomposed in a set of disjoint hypersurfaces corresponding to different integral values. Notice that if the system is time-dependent, the orbits may intersect for different times, although, for a fixed time t_0, the integrals must define a family of level curves of the phase space.

Let us briefly review some typical examples of isolating integrals. In general, up to three isolating integrals are found for all orbits under steady-state and axisymmetric potentials. Thus, by expressing the velocity components (V_1, V_2, V_3) in a Cartesian heliocentric coordinate system, with V_1

toward the Galactic centre, V_2 in the rotational direction, and V_3 perpendicular to Galactic plane, positive in the direction of the North Galactic pole, for a stationary potential $\mathcal{U}$ the energy integral can be written as

$$I_1 = V_1^2 + V_2^2 + V_3^2 + 2\mathcal{U}(\boldsymbol{r}). \tag{1.5}$$

The integral for the axial component of the angular momentum under an axisymmetric potential is expressed, in cylindrical coordinates $\boldsymbol{r} = (r, \theta, z)$, as

$$I_2 = rV_2. \tag{1.6}$$

Also, under a separable potential $\mathcal{U} = \mathcal{U}_1(r) + \mathcal{U}_2(z)$, which is valid near the Galactic plane, it is obtained a third integral, sometimes called Oort's integral,

$$I_3 = V_3^2 + 2\mathcal{U}_2(z). \tag{1.7}$$

Of course, any combination of above integrals is also conserved. Let us point out two very simple cases. For a fixed position in the Galaxy, the quantity

$$I_4 = V_1^2 + V_3^2 \tag{1.8}$$

is also an isolating integral. Similarly, a quadratic function

$$I_5 = V_1^2 + \alpha(V_2 - V_0)^2 + \beta V_3^2 \tag{1.9}$$

for any V_0, α and β constants or depending on the position, is also conserved. The later may be generalised, under appropriate hypotheses, to arbitrary quadratic functions of the peculiar velocity components, which justifies the generalised use of Gaussian type velocity distributions.

When some kinematic knowledge about the stellar system is available, such as that concerning the integrals of motion, if the density function f is already known, Eq. 1.2 may be interpreted according to the Jeans' inverse problem as a linear, non-homogeneous partial differential equation for $\mathcal{U}$. For example, the velocity distribution of some stellar groups can be assumed, after a transient period, of Maxwell type, Schwarzschild type, or ellipsoidal shaped (e.g., de Zeeuw & Lynden-Bell 1985). This viewpoint is a functional approach, which generally focuses on the study of a single stellar population (or may be used to define a statistical population) where the gravitational potential and the total stellar density N are related through the Newton-Poisson equation

$$\nabla_{\boldsymbol{r}}^2 \mathcal{U} = 4\pi G N.$$

However, self-consistent models that use the above equation are very limited, since unknown stellar populations –including gas and dark matter– do contribute also to the gravitational field.

In addition, it may be combined with a mixture model to get a more complete portrait of the velocity distribution (e.g., Cubarsi 1990, 1992). With this viewpoint, there is no need of the collisions term. On the contrary, it is assumed that there are sufficient collisions to keep the system in statistical equilibrium, according to the specific phase space density function or the particular integrals of motion. In other words, it is assumed that the phase space density function is invariant under the collisional operator $C(f, f)$. The idea comes from the original work on statistical dynamics (Chandrasekhar 1943), where the collisional term, accounting for diffusion and frictional processes, is exactly what is needed to conserve the energy of the whole system and leave the Maxwellian distribution invariant. Therefore, it is not surprising that Lynden-Bell (1967), in studying the equilibrium distribution achieved after a violent relaxation process, induced by rapid fluctuations of the gravitational field, obtains a similar smooth distribution function for a rotating elliptical system, which is quadratic as in Chandrasekhar (1942). Notice, however, that the former uses the statistical dynamics approach, while the latter in this case faces the problem from analytical dynamics. Other examples are described in Ogorodnikov (1965) when deriving the most probable phase distribution after an efficient relaxation mechanism.

If $\mathcal{P}(t, \boldsymbol{r}, \boldsymbol{V})$ is an isolating integral of motion, continuous and differentiable in its arguments, for any fixed time t_0, the equation $\mathcal{P}(t_0, \boldsymbol{r}, \boldsymbol{V}) = C$ must define a one-parameter family of five-dimensional surfaces filling all the six dimensional phase space $\Gamma_r \times \Gamma_V$, for all the possible values of the constant $C \in I_P$. If we assume that the phase density depends only on $\mathcal{P}$, that is $f(t, \boldsymbol{r}, \boldsymbol{V}) = f(\mathcal{P})$, then $f(\mathcal{P})$ is also an isolating integral of motion, which must define, for the same fixed time t_0, another uniparametric family of curves $f(\mathcal{P}) = K \in I_{f(P)}$, associated with the same set of hypersurfaces filling $\Gamma_r \times \Gamma_V$. To each level curve of the former family corresponds one, and only one, level curve of the later family, so that $K = f(C)$. Thus, we can assume that f is a diffeomorphism in the interior of its domain. Hence, the following inequality must be fulfilled,

$$\frac{df(\mathcal{P})}{d\mathcal{P}} \neq 0. \tag{1.10}$$

In other words, $f(\mathcal{P})$ is a strictly increasing or decreasing, smooth function of the argument in any open set within the interval I_P. This is a basic property used in Chapter 4 for the general solution of the closure problem.

A typical example of this situation is the generalised Schwarzschild distribution, with $\mathcal{P} = Q + \sigma$, where $Q = \boldsymbol{u}^{\mathrm{T}} \cdot \boldsymbol{A}_2 \cdot \boldsymbol{u}$ is a quadratic, positive definite form depending on the peculiar velocity $\boldsymbol{u}$, where the second-rank symmetric tensor $\boldsymbol{A}_2$ and the scalar function σ depend only on time and position. Then, owing to Eq. 1.10, we can express the collisionless Boltzmann equation in either of the following forms

$$\frac{Df(\mathcal{P})}{Dt} = \frac{df(\mathcal{P})}{d\mathcal{P}} \frac{D\mathcal{P}}{Dt} = 0 \iff \frac{D\mathcal{P}}{Dt} = 0. \tag{1.11}$$

For the generalised Schwarzschild distribution, Chandrasekhar (1942) obtained a system of twenty partial differential equations for $\boldsymbol{A}_2$, σ, the mean velocity, and the potential, which is equivalent to the collisionless Boltzmann equation.

1.4 Self-consistent models

For isolated stellar systems the symmetries of the potential and the stellar density can be investigated from the self-consistency hypothesis according to a variant of the Jeans inverse problem. When the density function or the isolating integrals of the star's motion are known, the collisionless Boltzmann equation allows to determine the potential or some properties of the potential such as symmetry properties. The variant consists in taking into account the Poisson equation by relating the potential to the stellar density and assuming that the gravitational potential generated by the stellar system satisfying the stationary collisionless Boltzmann equation is the unique origin of the stellar system force field (e.g., An et al. 2017, and references therein). Theoretically, from the isolating integrals the stellar density can be determined by integrating the distribution function in terms of either the velocities or the integrals themselves, arising the dependence of the distribution function on the potential and the space coordinates. Once established this functional dependence, several theorems about the symmetry of the solutions of elliptical partial differential equations (in particular for the Poisson equation) lead to particular symmetries for the potential and the stellar density, such as symmetry plane and axisymmetry, without the need of solving any differential equation. Nevertheless, the existence of such a joint solution is not guaranteed.

Formally, the Poisson equation acts as a mathematical shortcut to deduce such symmetries instead of deducing them from the collisionless Boltzmann equation by assuming that there is no external force that favours any direction of the symmetry axis and, therefore, the average behaviour of the stellar fluid is symmetric. In this sense, it is worth mentioning the work by Camm (1941).

Camm considers the distribution function depending on a linear combination of the three integrals I_1, I_2, and I_3 (i.e., he assumes the ellipsoidal hypothesis). He solves the stationary collisionless Boltzmann equation and obtains a plane of symmetry for the velocity ellipsoid and a potential symmetric with respect to this plane. This is obtained without using the hypothesis of self-consistency. He proves that the potential, in addition to be axisymmetric, is either: (a) spherically symmetric, i.e., the solution below Eq. (19), in which he is not interested; (b) separable in addition, viz., the expression below Eq. (22), not consistent with a finite system; (c) Eq. (23), depending on the latitude (actually on its absolute value); (d) symmetric about an arbitrary plane z=0, i.e., the general solution below Eq. (23), in the new coordinates, which are the roots of the quadratic equation. Without using the hypothesis of self-consistency, the solutions that Camm thinks of physical significance (a,c,d) satisfy both separability for the potential and symmetry plane. But what is more interesting is that when he adds the Poisson equation he finds that there is no mathematical solution satisfying such a joint solution. It could also occur that a possible solution was so simple that it was totally unrealistic. Therefore, one must be cautious in using the self-consistency model.

Nevertheless, Camm forgets that, just as the ellipsoidal hypothesis is not valid for the whole system, neither the potential consistent with this model will be valid for the whole system. That is to say, in a certain region of the stellar system the potential, or its dominant term, can behave, for example, as the one associated to a quasi-elastic field of force (Ogorodnikov 1965), and therefore be separable in addition. The harmonic potential, according to Poisson equation, is the one created within a homogeneous spheroid. It would not be logical to pretend that such a model is extensible to an infinite galactic system but, in the central part of the system, the potential due to the galactic halo can certainly be modelled that way.

1.5 Stellar statistics

For fixed values of time t and position $\boldsymbol{r}$, the macroscopic properties of a stellar system can be described from the moments of the distribution, which provide indirect information on the phase-space density function $f(t, \boldsymbol{r}, \boldsymbol{V})$.

It is well-known that the first moments, accounting for the mean, give the more elementary property of the distribution; the second central moments describe how much the distribution is spread around the mean; the third moments describe distribution asymmetries like the skewness; the fourth moments are used to quantify how peaked the distribution is; and so forth (e.g., Stuart & Ord 1987).

In particular, the stellar density is given by

$$N(t, \boldsymbol{r}) = \int_{\Gamma_V} f(t, \boldsymbol{r}, \boldsymbol{V})\, d\boldsymbol{V} \tag{1.12}$$

and the stellar mean velocity, or velocity of the centroid, is

$$\boldsymbol{v}(t, \boldsymbol{r}) = \frac{1}{N(t, \boldsymbol{r})} \int_{\Gamma_V} \boldsymbol{V}\, f(t, \boldsymbol{r}, \boldsymbol{V})\, d\boldsymbol{V}. \tag{1.13}$$

The symmetric tensor of the n-order, non-centred trivariate moments is obtained from the expected value

$$\boldsymbol{m}_n(t, \boldsymbol{r}) = \langle (\boldsymbol{V})^n \rangle \equiv \frac{1}{N(t, \boldsymbol{r})} \int_{\Gamma_V} (\boldsymbol{V})^n\, f(t, \boldsymbol{r}, \boldsymbol{V})\, d\boldsymbol{V}, \quad n \geq 0 \tag{1.14}$$

where $(\cdot)^n$ stands for the n-tensor power. The tensor $\boldsymbol{m}_n$ then has $\binom{n+2}{2}$ different elements according to the expression

$$m_{i_1 i_2 \ldots i_n} = \langle V_{i_1} V_{i_2} \ldots V_{i_n} \rangle, \tag{1.15}$$

so that the indices belong to the set $\{1, 2, 3\}$, depending on the velocity component. Sometimes, instead of the component notation, namely Latin indices, it is also used a notation to make the velocity powers explicit, namely Greek indices, according to

$$m_{\alpha\beta\gamma} = \langle V_1^\alpha V_2^\beta V_3^\gamma \rangle. \tag{1.16}$$

Obviously, $\boldsymbol{m}_0 = 1$ and $\boldsymbol{m}_1 = \boldsymbol{v}(t, \boldsymbol{r})$, which is the mean velocity, or velocity of the centroid.

In a similar way, the symmetric tensor of the n-order centred moments is obtained by working from the peculiar velocity

$$\boldsymbol{u} = \boldsymbol{V} - \boldsymbol{v}(t, \boldsymbol{r}), \tag{1.17}$$

as the expected value

$$\boldsymbol{\mu}_n(t, \boldsymbol{r}) = \frac{1}{N(t, \boldsymbol{r})} \int_{\Gamma_V} (\boldsymbol{V} - \boldsymbol{v}(t, \boldsymbol{r}))^n\, f(t, \boldsymbol{r}, \boldsymbol{V})\, d\boldsymbol{V}, \quad n \geq 0, \tag{1.18}$$

with elements

$$\mu_{i_1 i_2 \ldots i_n} = \langle u_{i_1} u_{i_2} \ldots u_{i_n} \rangle. \tag{1.19}$$

In this case, $\boldsymbol{\mu}_0 = 1$ and $\boldsymbol{\mu}_1 = \mathbf{0}$. The second central moment $\boldsymbol{\mu}_2$ is also known as covariance matrix.

The second moment tensors, either centred or non-centred, are symmetric and positive-definite matrices, hence are diagonalizable with positive eigenvalues. When all the eigenvalues are equal, we say the tensor is isotropic. If an eigenvalue does not depend on $\boldsymbol{r}$, we say it is isothermal in the direction of the corresponding eigenvector.

The tensor of the central moments is related to the tensor of temperatures from the kinetic theory of gases, while the tensor of pressures is given by

$$\boldsymbol{P}_n = N\, \boldsymbol{\mu}_n. \tag{1.20}$$

Hereafter, when studying the velocity dependence of the distribution function from a statistical viewpoint, the variables of time and position are omitted, although they might be used in the framework of a dynamical model for the whole phase-space distribution function.

Ellipsoidal distributions, such as the Schwarzschild distribution, can be described in terms of their central second moments μ_{ij}, which sometimes are written with Latin indices, such as $\sigma_{ij}^2 = \langle V_i V_j \rangle - \langle V_i \rangle \langle V_j \rangle$ (Binney & Tremaine 2008). However, in other standard astronomy reference books, the Greek index notation is used (Gilmore, King & van der Kruit 1989), in particular when the velocity variables are expressed in the (U, V, W) coordinate system (without subindices), where the n-th moments $m_{\alpha\beta\gamma}$ satisfy $\alpha + \beta + \gamma = n$. The second central moments account for the shape and orientation of the velocity ellipsoid and for the variance σ_l^2 of the velocity distribution function in an arbitrary direction l of the peculiar velocity space. According to the coordinate system, if c_1 , c_2, and c_3 are the corresponding direction cosines, we have

$$\sigma_l^2 = \langle (c_1 u_1 + c_2 u_2 + c_3 u_3)^2 \rangle = \sum_{i,j} c_i c_j \mu_{ij}; \; i, j \in \{1, 2, 3\}. \tag{1.21}$$

The symmetric tensor $\boldsymbol{\mu}_2^{-1}$ (inverse of the second central moments $\boldsymbol{\mu}_2$) is then associated with the peculiar velocity ellipsoid

$$\boldsymbol{u}^{\mathrm{T}} \cdot \boldsymbol{\mu}_2^{-1} \cdot \boldsymbol{u} = 1, \tag{1.22}$$

so that the velocity dispersions σ_1 , σ_2, and σ_3 are the semiaxes of the ellipsoid that refers to the same coordinate axes.

Usually the direction u_1 is taken as the radial direction, having the greater velocity dispersion. Then, the major semiaxis of the velocity ellipsoid has a direction close to that of the location vector of the centroid. The angle of such a deviation is referred to as vertex deviation. The tilt of the velocity ellipsoid is also significant. If one of the principal semiaxes points to the Galactic centre, then there is no tilt, if not, the tilt is the angle of such a deviation. A precise definition of vertex deviation and tilt of the velocity ellipsoid is given in Appendix G.2).

For these distributions, higher order central moments can be computed depending on the second ones; in other words, they cannot take arbitrary values, as shown in Appendix C. However, for arbitrary distributions, the variances and the velocity ellipsoid are meaningless, unless they could be used as a Gaussian approximation. Similarly, the skewness and the kurtosis are also meaningless for multivariate distributions far from Gaussian, which have to be qualitatively described from moments of order higher than two, up to a sufficient degree of approximation of the basic distribution trends.

1.6 Velocity moments

In the very beginning of the book, Chandrasekhar (1942) outlines the appropriate conditions to define a unique local standard of rest (LSR) for describing the motions in a given relatively small volume of the Galaxy. The conditions are related to a continuous estimation of the centroid[2] velocity within this volume and to a slow varying distribution function, which could be referred to as regularity conditions. He concludes that the stellar systems can be divided into those for which the notion of LSR (and, by extension, higher velocity moments) is of significance and those for which it is not. Among the latter we can mention the systems dominated by a phase mixing process (e.g., Binney & Tremaine 2008), for which a macroscopic, coarse-grained distribution function may be defined in contrast with the true, fine-grained distribution function, although the coarse-grained distribution function of a mixing system would not satisfy the collisionless Boltzmann equation. We shall focus on the first class of stellar systems, for which the mean velocity and similar statistics are meaningful.

[2]The centroid of motion corresponds to the mean velocity of the stars in a small volume of the Galaxy and is used as synonymous of LSR. It is not exactly the same as the centre of mass of this volume, since we know little about the masses of individual stars. However, for the whole Galaxy, the galactic standard of rest is identified with the centre of gravity and not with the centroid of motion of all stars.

Let us remember that for the Galaxy and, in general, for stellar systems larger than globular clusters, the forces acting on a star can exclusively be associated with a mean gravitational field, by neglecting the random forces due to stellar encounters. In the solar neighbourhood, assuming that the Galaxy has reached an equilibrium configuration, the potential is usually taken as explicitly time-independent. Then, the Hamiltonian flow possesses the energy integral, is always *nonergodic* and, therefore, *nonmixing* (e.g., Arnold & Avez 1968).

Thus, in order to introduce the kinematic statistics into the collisionless Boltzmann equation, Eq. 1.2 is multiplied by the n-tensor power of the star velocity and then integrated over the whole velocity space,

$$\int_{\Gamma_V} (\boldsymbol{V})^n \, \frac{Df}{Dt} \, d\boldsymbol{V} = (\boldsymbol{0})^n, \quad n \geq 0 \tag{1.23}$$

where in the integration process the following boundary conditions are assumed because there are no stars with velocity beyond Γ_V

$$\boldsymbol{V} \to \partial\Gamma_V \Longrightarrow (\boldsymbol{V})^n f(t, \boldsymbol{r}, \boldsymbol{V}) \to (\boldsymbol{0})^n, \; n \geq 0. \tag{1.24}$$

It is always assumed that the foregoing integrals do exist, as those of the velocity moments.

For each value of n, the tensor equation Eq. 1.23 leads to the n-order stellar hydrodynamic equation, which provides us with a conservation or transfer law *along the centroid trajectory*. The most basic cases are the continuity equation, for $n = 0$, which stands for mass conservation, and the momentum conservation equation, for $n = 1$.

However, the methodology of most books on galactic dynamics which devote a chapter to obtain or discuss the stellar hydrodynamic equations (e.g., Chandrasekhar 1942, Kurth 1957, Ogorodnikov 1965, Mihalas 1968, Binney & Tremaine 2008) is to integrate Eq. 1.23 –for $n = 0$ and $n = 1$– over the absolute, non-peculiar velocities, leading to equations involving the absolute moments of the velocity distribution, and afterwards, in order to give a physical interpretation of each equation, the total moments are explicitly written in function of the central moments. Such a procedure is appropriate for the lowest order equations, but it is not adequate for an arbitrary n-order equation.

Chapter 2

Stellar hydrodynamic equations

2.1 Comoving-frame equations

Let us write the collisionless Boltzmann equation, Eq. 1.2, in terms of the stellar mean velocity, Eq. 1.13, and of the peculiar velocities, Eq. 1.17, by expressing the phase space density function f in the form

$$\phi(t, \boldsymbol{r}, \boldsymbol{u}) = f(t, \boldsymbol{r}, \boldsymbol{u} + \boldsymbol{v}(t, \boldsymbol{r})) \tag{2.1}$$

where t, $\boldsymbol{r}$ and $\boldsymbol{u}$ are independent variables. Hence, the derivatives with respect to these variables are

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{\partial \phi}{\partial t} + \frac{\partial \boldsymbol{u}}{\partial t} \cdot \nabla_{\boldsymbol{u}} \phi = \frac{\partial \phi}{\partial t} - \frac{\partial \boldsymbol{v}}{\partial t} \cdot \nabla_{\boldsymbol{u}} \phi, \\ \nabla_{\boldsymbol{r}} f &= \nabla_{\boldsymbol{r}} \phi + \nabla_{\boldsymbol{r}} \boldsymbol{u} \cdot \nabla_{\boldsymbol{u}} \phi = \nabla_{\boldsymbol{r}} \phi - \nabla_{\boldsymbol{r}} \boldsymbol{v} \cdot \nabla_{\boldsymbol{u}} \phi, \\ \nabla_{\boldsymbol{V}} f &= \nabla_{\boldsymbol{V}} \boldsymbol{u} \cdot \nabla_{\boldsymbol{u}} \phi = \nabla_{\boldsymbol{u}} \phi. \end{aligned} \tag{2.2}$$

Then Eq. 1.2 becomes

$$\frac{\partial \phi}{\partial t} - \frac{\partial \boldsymbol{v}}{\partial t} \cdot \nabla_{\boldsymbol{u}} \phi + (\boldsymbol{u} + \boldsymbol{v}) \cdot (\nabla_{\boldsymbol{r}} \phi - \nabla_{\boldsymbol{r}} \boldsymbol{v} \cdot \nabla_{\boldsymbol{u}} \phi) - \nabla_{\boldsymbol{r}} \boldsymbol{\mathcal{U}} \cdot \nabla_{\boldsymbol{u}} \phi = 0 \tag{2.3}$$

so that, by reorganising terms, it yields

$$\frac{\partial\phi}{\partial t}+\boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}\phi-\left(\frac{\partial\boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}\boldsymbol{v}\right)\cdot\nabla_{\boldsymbol{u}}\phi+\boldsymbol{u}\cdot(\nabla_{\boldsymbol{r}}\phi-\nabla_{\boldsymbol{r}}\boldsymbol{v}\cdot\nabla_{\boldsymbol{u}}\phi)-\nabla_{\boldsymbol{r}}\boldsymbol{\mathcal{U}}\cdot\nabla_{\boldsymbol{u}}\phi=0. \tag{2.4}$$

To simplify the notation, we use the material derivative (also called substantial derivative) associated with the motion of the centroid,

$$\frac{d}{dt}(\cdot)=\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}\right)(\cdot)\,. \tag{2.5}$$

Since $\boldsymbol{r}$ and $\boldsymbol{u}$ are independent variables, we take into account the identity

$$\boldsymbol{u}\cdot\nabla_{\boldsymbol{r}}\phi=\nabla_{\boldsymbol{r}}\cdot(\boldsymbol{u}\,\phi) \tag{2.6}$$

and we also consider the following equality[1]

$$\boldsymbol{u}\cdot\nabla_{\boldsymbol{r}}\boldsymbol{v}\cdot\nabla_{\boldsymbol{u}}\phi=\nabla_{\boldsymbol{r}}\boldsymbol{v}:(\boldsymbol{u}\otimes\nabla_{\boldsymbol{u}}\phi) \tag{2.7}$$

where each dot represents an inner product, and $\otimes$ a tensor product[2]. Notice that the colon stands for the dot products $\nabla_{\boldsymbol{r}}$ with $\boldsymbol{u}$, and $\boldsymbol{v}$ with $\nabla_{\boldsymbol{u}}$, respectively.

Hence, Eq. 2.4 may be written as follows

$$\frac{d\phi}{dt}-\left(\frac{d\boldsymbol{v}}{dt}+\nabla_{\boldsymbol{r}}\boldsymbol{\mathcal{U}}\right)\cdot\nabla_{\boldsymbol{u}}\phi+\nabla_{\boldsymbol{r}}\cdot(\boldsymbol{u}\phi)-\nabla_{\boldsymbol{r}}\boldsymbol{v}:(\boldsymbol{u}\otimes\nabla_{\boldsymbol{u}}\phi)=0. \tag{2.8}$$

We take now the tensor product of the foregoing equation with the n-tensor power of the peculiar velocity $(\boldsymbol{u})^n$,

$$\begin{aligned}(\boldsymbol{u})^n\,\frac{d\phi}{dt}-(\boldsymbol{u})^n\otimes\left[\left(\frac{d\boldsymbol{v}}{dt}+\nabla_{\boldsymbol{r}}\boldsymbol{\mathcal{U}}\right)\cdot\nabla_{\boldsymbol{u}}\phi\right]+\\ +\nabla_{\boldsymbol{r}}\cdot\left[(\boldsymbol{u})^{n+1}\phi\right]-\nabla_{\boldsymbol{r}}\boldsymbol{v}:\left[(\boldsymbol{u})^{n+1}\otimes\nabla_{\boldsymbol{u}}\phi\right]=(\boldsymbol{0})^n\end{aligned} \tag{2.9}$$

[1]In component notation the equality can be written as $u_i\dfrac{\partial v_j}{\partial r_i}\dfrac{\partial\phi}{\partial u_j}=\dfrac{\partial v_j}{\partial r_i}\left(u_i\dfrac{\partial\phi}{\partial u_j}\right)$, where Einstein's summation criterion for repeated indices is applied.

[2]The notation used for nabla operators is the usual one. If $\boldsymbol{x}$ is a vector variable and $\boldsymbol{F}_n(\boldsymbol{x})$ a n-rank symmetric tensor field, then for $n\geq 1$ the divergence $\nabla_{\boldsymbol{x}}\cdot\boldsymbol{F}_n$ is, in components, $\frac{\partial}{\partial x_{i_1}}F_{i_1\ldots i_n}$, while for $n\geq 0$, $\nabla_{\boldsymbol{x}}\boldsymbol{F}_n$ is used instead of $\nabla_{\boldsymbol{x}}\otimes\boldsymbol{F}_n$ to represent the gradient $\frac{\partial}{\partial x_{i_1}}F_{i_2\ldots i_{n+1}}$.

and the resulting equation is then integrated over the peculiar velocities space Γ_u, where the factors depending only on $\boldsymbol{r}$ and t are left out of the integrals. Thus, we obtain

$$\frac{d}{dt}\int_{\Gamma_u}(\boldsymbol{u})^n\,\phi\,d\boldsymbol{u}-\left(\frac{d\boldsymbol{v}}{dt}+\nabla_{\boldsymbol{r}}\mathcal{U}\right)\cdot\int_{\Gamma_u}(\boldsymbol{u})^n\otimes\nabla_{\boldsymbol{u}}\phi\,d\boldsymbol{u}+ \\ +\nabla_{\boldsymbol{r}}\cdot\int_{\Gamma_u}(\boldsymbol{u})^{n+1}\,\phi\,d\boldsymbol{u}-\nabla_{\boldsymbol{r}}\boldsymbol{v}:\int_{\Gamma_u}(\boldsymbol{u})^{n+1}\otimes\nabla_{\boldsymbol{u}}\phi\,d\boldsymbol{u}=(\boldsymbol{0})^n. \tag{2.10}$$

The first and third terms of the above relationship are directly expressed in function of the pressures, according to Eq. 1.14 and Eq. 1.20. Instead, for the other terms an auxiliary tensor may be defined as follows

$$\boldsymbol{Q}_{n+1}=-\int_{\Gamma_u}(\boldsymbol{u})^n\otimes\nabla_{\boldsymbol{u}}\phi\,d\boldsymbol{u},\ n\geq 0 \tag{2.11}$$

so that Eq. 2.10 may be rewritten in a more compact notation,

$$\frac{d\boldsymbol{P}_n}{dt}+\left(\frac{d\boldsymbol{v}}{dt}+\nabla_{\boldsymbol{r}}\mathcal{U}\right)\cdot\boldsymbol{Q}_{n+1}+\nabla_{\boldsymbol{r}}\cdot\boldsymbol{P}_{n+1}+\nabla_{\boldsymbol{r}}\boldsymbol{v}:\boldsymbol{Q}_{n+2}=(\boldsymbol{0})^n. \tag{2.12}$$

However, the tensors $\boldsymbol{Q}_n$ are not directly computable in their current form.

2.2 Conservation of pressures

The next step is to write the general hydrodynamic equation Eq. 2.12 explicitly depending on the pressures. Hence we shall find out how the tensors $\boldsymbol{Q}_n$ can be expressed in terms of the pressures $\boldsymbol{P}_n$.

Let us note a particular case of Eq. 2.11. For $n=0$, bearing in mind the boundary condition Eq. 1.24, we get

$$\boldsymbol{Q}_1=-\int_{\Gamma_u}\nabla_{\boldsymbol{u}}\phi\,d\boldsymbol{u}=\phi|_{\boldsymbol{u}}=\boldsymbol{0}. \tag{2.13}$$

For $n=1$, the tensor product $(\boldsymbol{u})^n\otimes\nabla_{\boldsymbol{u}}\phi$ within the integral of Eq. 2.11 verifies, in components,

$$u_i\,\frac{\partial\phi}{\partial u_j}=\frac{\partial(u_i\phi)}{\partial u_j}-\delta_{ij}\,\phi, \tag{2.14}$$

being δ_{ij} the Kronecker delta, and for $n \geq 2$,

$$u_{i_1} \dots u_{i_n} \frac{\partial \phi}{\partial u_{i_{n+1}}} = \frac{\partial (u_{i_1} \dots u_{i_n} \phi)}{\partial u_{i_{n+1}}} - \Big(\delta_{i_1 i_{n+1}} u_{i_2} \dots u_{i_n} + \dots + \\ + \delta_{i_j i_{n+1}} u_{i_1} \dots \widehat{u_{i_j}} \dots u_{i_n} + \dots + \delta_{i_n i_{n+1}} u_{i_1} \dots u_{i_{n-1}}\Big) \phi \tag{2.15}$$

where the hat remarks the omitted factors.

Then, the tensor $\boldsymbol{Q}_{n+1}$ can be evaluated by integrating Eq. 2.14 and Eq. 2.15. The conditions of Eq. 1.24 are once more applied over the integration boundary, so that the first term on the right-hand side of Eq. 2.15, when integrating over $u_{i_{n+1}}$, yields

$$\int_{u_{i_{n+1}}} \frac{\partial (u_{i_1} \dots u_{i_n} \phi)}{\partial u_{i_{n+1}}} du_{i_{n+1}} = u_{i_1} \dots u_{i_n} \phi|_{u_{i_{n+1}}} = 0. \tag{2.16}$$

Hence, the tensor $\boldsymbol{Q}_{n+1}$ is obtained by integrating only the remaining terms, and by taking into account Eq. 1.19 and Eq. 1.20.

Thus, for $n = 1$ we are led to

$$(\boldsymbol{Q}_2)_{ij} = \delta_{ij} \boldsymbol{P}_0 \tag{2.17}$$

and for $n \geq 2$, we get the following expression depending on the pressures,

$$(\boldsymbol{Q}_{n+1})_{i_1 \dots i_{n+1}} = \delta_{i_1 i_{n+1}} P_{i_2 \dots i_n} + \dots + \delta_{i_j i_{n+1}} P_{i_1 \dots \widehat{i_j} \dots i_n} + \dots + \delta_{i_n i_{n+1}} P_{i_1 \dots i_{n-1}}. \tag{2.18}$$

The foregoing relationships will be used to write both terms in Eq. 2.12, which involve the tensor $\boldsymbol{Q}_{n+1}$. One of the terms contains a single dot product of this tensor with a vector, namely $\boldsymbol{a} \cdot \boldsymbol{Q}_{n+1}$. Hence, by applying Eq. 2.18, we get

$$(\boldsymbol{a} \cdot \boldsymbol{Q}_{n+1})_{i_1 \dots i_n} = \\ = a_{i_{n+1}} \Big(\delta_{i_1 i_{n+1}} P_{i_2 \dots i_n} + \dots + \delta_{i_j i_{n+1}} P_{i_1 \dots \widehat{i_j} \dots i_n} + \dots + \delta_{i_n i_{n+1}} P_{i_1 \dots i_{n-1}}\Big) = \\ = a_{i_1} P_{i_2 \dots i_n} + \dots + a_{i_j} P_{i_1 \dots \widehat{i_j} \dots i_n} + \dots + a_{i_n} P_{i_1 \dots i_{n-1}} \tag{2.19}$$

where Einstein's summation convention is used. The result is the symmetrised tensor product in regard to permutations of indices, $\mathcal{S}(\boldsymbol{a} \otimes \boldsymbol{P}_{n-1})$,