Math Outside the Classroom

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To Quanyi, Sherrie and Janice

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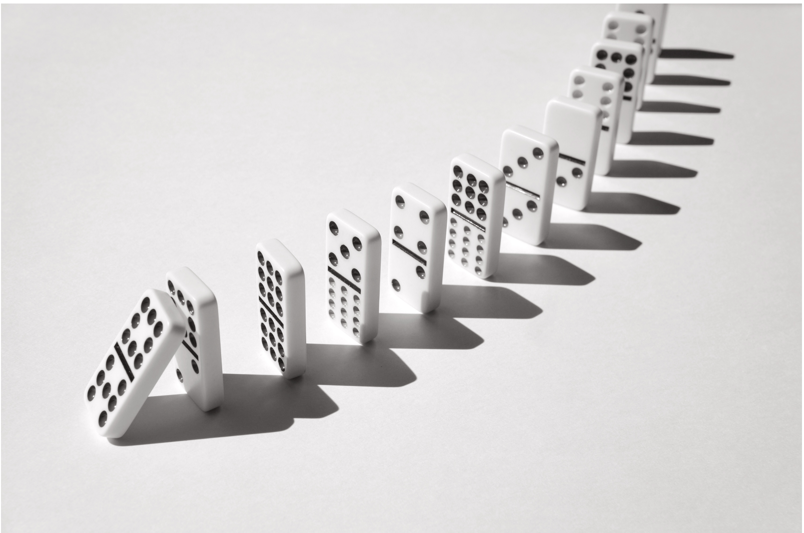
Introduction



Amy and Ben are playing dominoes after school. Amy aligns the tiles in a line and then triggers their cascading effect, while Ben seems engrossed in solving a puzzle.

“Today, we learned about Mathematical Induction. It’s similar to playing dominoes," says Amy.

“How so?" Ben inquires.

“Mathematical Induction allows us to prove that a statement holds true for all natural numbers, n. It involves two steps. First, we prove that the statement holds for the base case, which is typically $n = 1$. Then, we show that if the statement holds for $n = k$, it also holds for $n = k + 1$. In dominoes, this translates to ensuring that the first tile will fall, and if any tile falls, the next one will follow suit."

“Did you use induction for today’s math homework?" Ben asks.

“Yes, we were tasked with proving that the sum of the first n odd numbers is equal to n^2. The n-th odd number is $2n-1$, and we needed to prove the following:

$$1+3+5+\cdots+(2n-1)=n^2$$

for all n. The steps were as follows:

Step 1: $1=1^2$;

Step 2: Assume $1+3+5+\cdots+(2n-1)=n^2$. The $(n+1)$-th odd number is $2n+1$, so we have

$$1+3+5+\cdots+(2n-1)+(2n+1)=n^2+(2n+1)=(n+1)^2.$$

And that’s it."

“But, I was going to share something new," Ben says, before Amy cuts him off.

“I know, another new idea," Amy interjects with a smile.

Ben then arranges four rows of dominoes, each representing the numbers $1,3,5,$ and 7. He splits them into two parts and flips one over, before joining them to form a square (as shown in Figure 2).

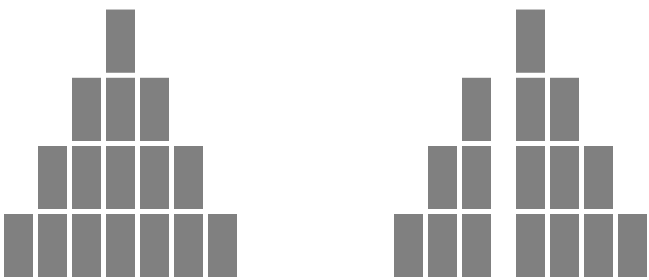


Figure 1: Sum of the first 4 odd numbers

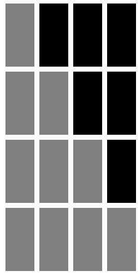


Figure 2: Rearrangement: a square of 4×4

“Look, The first n odd numbers can always be represented in this way." Ben explains.

“Can this process be explained algebraically?" asks Amy.

“I’ll try. We can simply divide every odd number into two parts:

$1 = 1+0,\ \ 3 = 2+1,\ \ 5 = 3+2\ , \cdots,\ 2n-1 = n+(n-1).$

This also cuts the original series into two series:

$$\begin{aligned} & 1+3+5+\cdots+(2n-1) \\ & \quad = (1+2+3+\cdots+n)+(0+1+2+\cdots+n-1) \\ & \quad = (1+2+3+\cdots+n)+(n-1+\cdots+2+1+0) \\ & \quad = (1+n-1)+(2+n-2)+\cdots+(n-1+1)+(n+0) \\ & \quad = n+n+\cdots+n \\ & \quad = n^2 \end{aligned}$$

This demonstrates that why we can arrange dominoes into a square."

“That sounds right, but you didn’t use mathematical induction." Amy reminds him of the teacher’s instructions.

Ben responds, “Do I have to? I’m not comfortable with induction because of an example that still bothers me $\cdots$"

“Wait a second, how? Could you explain?" Amy asks, surprised.

“Let me demonstrate how to use mathematical induction to prove a false statement: ‘all dominoes have the same number of pips.’ First, starting with one domino, it has a number. Now consider $k+1$ dominoes, assuming that any k dominoes have the same number of pips. If all dominoes are placed in a line, the assumption is that the first k dominoes have the same number of pips; similarly, the last k dominoes have the same number of pips. Because there must be a domino in both groups, all $k+1$ dominoes must have the same number of pips."

“Something must be wrong with your proof. Let’s figure out which part of using mathematical induction was incorrect."

~ * ~ * ~

Amy and Ben are two student characters in this book, who may be familiar to you as classmates, colleagues, or even yourself. Many students are like Amy, who follows her teacher’s instructions in class and completes her homework in the traditional manner. This is a highly efficient way of meeting the requirements of the school and class, and these students are often pleased with their academic performance. However, it is important for them to understand that the traditional education system only focuses on a limited aspect of mathematics, and that being a good student involves more than just getting good grades. It requires a deep understanding of the subject and the ability to apply it in real-world situations.

In the real world, students will encounter problems that are not covered in textbooks and will require them to think creatively and outside the box. After leaving school and entering the workforce, students are expected to solve real-world problems that require innovative thinking. One way to overcome the limitations of traditional education is to ask more questions and challenge your thinking. By asking “Are there any other options?" and “What would happen if I didn’t take this step?" students can explore different solutions to a problem and see which one is most effective. This questioning approach helps students to deepen their understanding of the subject.

On the other hand, a student like Ben, who is constantly seeking new ideas and pursuing what interests them, can sometimes face challenges in the classroom. They may not always follow the teacher’s instructions for problem-solving, which can lead to difficulty finding a solution. Additionally, some teachers may only award credit for solutions that strictly follow their guidelines. However, having a diverse range of perspectives should be encouraged, as it enriches the understanding of various concepts. Creativity is a crucial skill for students who plan to pursue careers in science or engineering where creativity and innovative problem-solving skills are highly valued, but it's more than just having a curious mind. These students must also learn how to effectively and efficiently achieve their goals. This can be accomplished by reading various sources beyond just textbooks and learning from others. It's important to avoid aimless searching without a clear objective.

For students who struggle with math, the subject can feel overwhelming and uninteresting. However, it's important to remember that there are ways to overcome these obstacles and find a renewed appreciation for math.

One approach is to seek out resources that are more accessible and cater to different learning styles. Reading books and articles, watching educational videos, and exploring online math learning communities can help you discover new ways of thinking about math and develop a deeper understanding of its concepts. Another way to appreciate the beauty of math is by exploring its real-world applications. Math is used in fields such as science, engineering, and finance, and understanding its role in these areas can help you see its relevance and importance. You must be willing to explore it in new and meaningful ways. With persistence, creativity, and a willingness to learn, you can overcome your challenges and discover the excitement and wonder of this fascinating subject.

This book aims to provide a different approach to learning mathematics, one that is designed for all types of readers, including those who have already graduated but want to improve their problem-solving skills. The author intends to offer readers a new perspective on the world of mathematics by emphasizing the interdependence of various mathematical concepts, including those students may not have encountered in the classroom, and providing alternative approaches to problem-solving. With this book, readers will have the opportunity to learn new concepts and have fun while doing so, as they discover the exciting and interesting world of mathematics.

Albert Einstein once said, "Education is not the learning of facts; it is rather the training of the mind to think." This book aligns with this philosophy, as it strives to assist students in developing their problem-solving skills and strengthening their grasp of mathematics.

Chapter 1
Limits

“Have you heard the story of the Tortoise and the Hare?" Ben asks Amy one day.

“Of course, the hare lost " Amy responds.

“Do you know why the hare was defeated? "

“Because the hare was napping when he reached the halfway point "

“It’s more of a joke than a story, in my opinion. Achilles and the Tortoise is an ancient Greek story that I enjoy. In the story, the quick-footed hero lost the race against the tortoise. No, I mean the tortoise presented a logical argument that Achilles was unable to defeat or refute.

“What was the argument? Did they expect Achilles to take a nap?"

“No, let me tell you a story." Ben then tells his version of the ancient legend.

> *Achilles and a tortoise engaged in a race. Achilles realized that even if he won, people would not think he did so fairly. So he proposed to the tortoise:*
>
> *“Because you are slow, I will let you run first, and I will start after you have run 10 meters."*
>
> *“Well, before you begin," the tortoise said, “I will let you know that you cannot catch me."*
>
> *“ Why?"*
>
> *The tortoise explained his reason slowly, “Assume you move 10 times as quickly as I do. When you start, I am already 10 meters ahead. When you finish these 10 meters, I will have finished another 1 meter. Then you cover this 1 meter gap, but by then I have advanced another 1/10 meter $\cdots$ The point is that there is always a gap between you and me. I am always right in*

front of you. Got it?"
Achilles: " ⋯ "

"I see. The story creates a sequence of positive numbers that approaches the limit, zero," Amy says.

1.1 Limits and Sequences

What is a limit? This is a critical concept in mathematics. Following Ben's story, the distance between the tortoise and Achilles changes and can be represented as a sequence as follows:

$$10,\ 1,\ \frac{1}{10},\ \frac{1}{100}\ \cdots \tag{1.1}$$

You can write this sequence for as long as you want and can get the number as close to zero as you like. This sequence is said to be approaching zero, or its limit is zero. Here is another example:

$$\frac{1}{3},\ \frac{2}{5},\ \frac{3}{7},\ \cdots,\ \frac{n}{2n+1},\ \cdots \tag{1.2}$$

It is not difficult to see intuitively that the limit of this sequence is $\frac{1}{2}$.

In general, a sequence is denoted as

$$(a_n) = a_1, a_2, \cdots \tag{1.3}$$

The fact that "the limit of a sequence (a_n) is L," simply denoted by $a_n \to L$, means that the distance between the terms of this sequence and L, $|a_n - L|$, can be arbitrarily close to zero. How can something be described as "arbitrarily close" ? One way to put it is: "no matter how small the positive number you choose, $|a_n - L|$ will always be smaller than your number whenever n is large enough. In mathematics, people use so-called "$\varepsilon - N$ definition" to describe the limit of a sequence. Here is this formal definition.

Definition 1.1.1. *A sequence (a_n) has the limit L as n goes to infinity, or $\lim_{n\to\infty} a_n = L$, which means that for each real number $\varepsilon > 0$, there is an integer N, such that if $n > N$, then $|a_n - L| < \varepsilon$.*

Students learning the "$\varepsilon - N$ definition" for the first time may find it is unfamiliar. We will demonstrate how it works with an example and then extend on the concept of the limit.

Example 1.1.1.

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \tag{1.4}$$

Proof. By definition, we only need to show that for each $\varepsilon > 0$, there exists an integer N such that whenever $n > N$, we have $\left|\frac{n}{n+1} - 1\right| < \varepsilon$. Since $\frac{n}{n+1} < 1$, the last inequality is equivalent to $1 - \frac{n}{n+1} < \varepsilon$, or $n > \frac{1}{\varepsilon} - 1$. It is now clear that if N is an integer such that $N > \frac{1}{\varepsilon} - 1$, then for each $n > N$, we have $\left|\frac{n}{n+1} - 1\right| < \varepsilon$. This proves (1.4). □

In the case of the sequence in (1.1), $a_n = 10^{2-n}$ and $L = 0$. The "$\varepsilon - N$ definition" for this limit, $\lim_{n\to\infty} a_n = 0$, is that for each $\varepsilon > 0$, there is an integer N such that if $n > N$, then $|10^{2-n} - 0| < \varepsilon$. In fact, the last inequality is equivalent to $n > 2 - \log_{10}\varepsilon$. So choose any integer $N > 2 - \log_{10}\varepsilon$, we have that if $n > N$, then $|10^{2-n} - 0| < \varepsilon$. In this case, $a_n \neq L$ for all n, but this is not a necessary condition for defining the limit. Readers can try to prove the limit of sequence (1.2) using an $\varepsilon - N$ definition.

Example 1.1.2.

$$\lim_{n\to\infty} \cos n\pi \text{ does not exist}$$

In fact, $(a_n) = (\cos n\pi)$ is a simple sequence:

$$-1,\ 1,\ -1,\ 1,\ \cdots,\ (-1)^n,\ \cdots$$

Clearly, it will not approach to any number as $n \to \infty$. How can we state this fact formally? By definition, if its limit, say L, exists, then for any $\varepsilon > 0$, there exists an N, $|a_n - L| < \varepsilon$ whenever $n > N$. What we need to show is there is a number ε, no matter how large N is, we can always find $n > N$ such that $|a_n - L| \geq \varepsilon$.

Proof. Let $\varepsilon = 1/2$. If the limit exists, say L, then there is N, such that $|a_n - L| < \frac{1}{2}$ whenever $n > N$. If $a_n = 1$, then $a_{n+1} = -1$. We have

$$2 = a_n - a_{n+1} = |a_n - L + L - a_{n+1}| \leq |a_n - L| + |L - a_{n+1}| < \frac{1}{2} + \frac{1}{2} = 1$$

This implies that we cannot have both $|a_n - L| < \varepsilon$ and $|a_{n+1} - L| < \varepsilon$. It is sufficient to conclude that $\lim_{n\to\infty} \cos n\pi$ does not exist. □

Consider that we have a sequence (a_n). If (a_n) converges to L, then its "tail" $T_m = (a_m, a_{m+1}, \cdots)$ also converges to L, and vice versa. In other words, removing any finite terms from (a_n) will not change its convergence feature. That might be translated as "What has happened does not matter, what counts is how the future looks." Another common phrase is "a_n is getting closer to L as n gets bigger and bigger." There are upper and lower bounds for each tail T_m, and the gap between two bounds goes to zero as m goes to infinity.

Definition 1.1.2. *Let (a_n) be an infinite sequence. A tail T_m is its subsequence $(a_m, a_{m+1}, \cdots)$. A channel $C(L, \varepsilon)$ is defined as a pair of horizontal lines $y = L - \varepsilon$ and $y = L + \varepsilon$. It is called a bound channel of (a_n) if it "contains" at least one tail of (a_n). In other words, there is $m > 0$ such that $L - \varepsilon \leq a_n \leq L + \varepsilon$ for all $n \geq m$.*

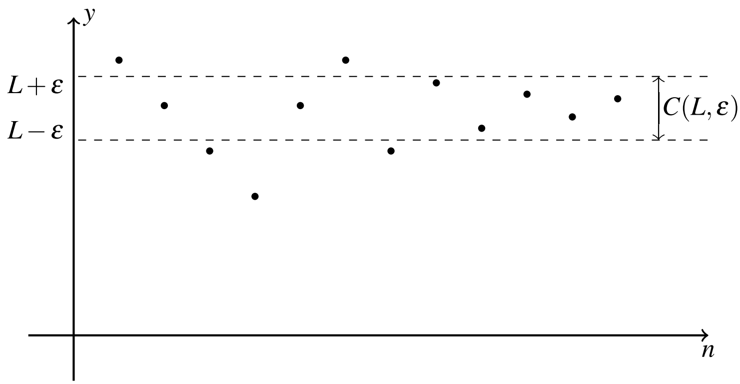


Figure 1.1: A bound channel of (a_n)

There are some equivalent characterizations for a convergent sequence. We list some of them in the following lemma.

Lemma 1.1.3. *Let (a_n) be an infinite sequence. The following conditions are equivalent:*

(1). (a_n) converges to L;

(2). Any tail of (a_n) converges to L;

(3). There is an infinite sequence of bound channels of (a_n):

$$C(L, 1),\ C(L, 1/2),\ C(L, 1/3),\ \cdots,\ C(L, 1/m),\ \cdots$$

Proof. (1) and (2) are clearly equivalent. We will show the equivalence of (1) and (3). Assume (a_n) converges to L. For $\varepsilon = \dfrac{1}{m}$, there is $N > 0$, such that $|a_n - L| < \dfrac{1}{m}$ for all $n > N$. That is, channel $C\left(L, \dfrac{1}{m}\right)$ contains tail T_N and thus it is a bound channel. This proves that (1) implies (3). Assume (3) holds. For any $\varepsilon > 0$, we can find a natural number m such that $\dfrac{1}{m} < \varepsilon$. Since $C\left(L, \dfrac{1}{m}\right)$ is a bound channel by (3), it contains a tail, say T_N. That is, for all $n > N$, we have $L - \varepsilon \leq a_n \leq L + \varepsilon$, or $|a_n - L| < \varepsilon$. This proves that (a_n) converges to L. □

In Lemma 1.1.3, the way one chooses the bound channels is not important. We can replace them with

$$C(L, \varepsilon_1),\ C(L, \varepsilon_2),\ \cdots,\ C(L, \varepsilon_m),\ \cdots$$

where $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m, \cdots)$ can be any positive-termed decreasing sequence and $\varepsilon_m \to 0$.

What was the issue with the Tortoise's argument? The trick here is that in Ben's story, he artificially records the distance between the two into an infinite sequence and does not allow it exceed the limit. The sequence is well-defined but he overlooks an important factor in the real problem — time. If the time required to travel each distance in the sequence has a lower bound, then it would take forever, since the sequence has infinite terms. However, in this race, the time required to complete each term in the sequence can be listed as: 1 second, 1/10 second, 1/100 second $\cdots$ (assume Achilles ran 10 meters per second.) We add all of these time periods to see how long it takes:

$$T = 1 + \frac{1}{10} + \frac{1}{100} + \cdots + \frac{1}{10^{n-1}} + \cdots$$

This is a geometric series with the common ratio $1/10$. We will discuss it later. The sum of this series is

$$T = \frac{1}{1 - \frac{1}{10}} = \frac{10}{9}$$

As a result, Achilles only needs $10/9 (\approx 1.11)$ seconds to catch the tortoise. Ben's statement is correct only from 0 to 10/9 seconds. Then, from this point, it is essentially the start of a new game — the tortoise and Achilles are both starting from the same line. The outcome is obvious.

Exercises

1. Using "$\varepsilon - N$ definition" prove
$$\lim_{n\to\infty} \frac{\sin n + n}{n^2} = 0$$

2. Using "$\varepsilon - N$ definition" show that a sequence can have at most one limit.

3. Prove that
$$\lim_{n\to\infty} a^n = \begin{cases} \infty, & \text{if } a > 1; \\ 1, & \text{if } a = 1; \\ 0, & \text{if } |a| < 1; \end{cases}$$

(What is the correct way to say $a_n \to \infty$ as $n \to \infty$?)

1.2 Limits of Functions

An infinite sequence $\{a_n\}$ may have a limit as n goes to infinity. Similarly, the limit of a function $f(x)$ as x goes to infinity, can be defined. For example,
$$\lim_{x\to\infty} \frac{x}{x+1} = 1$$
We can make the same argument as in the previous section. However, the limit of a function may also exist as x approaches a number. We say that $f(x)$ is arbitrarily close to L as x approaches a number. The formal definition is as follows.

Definition 1.2.1. *Let $f(x)$ be a function that is defined on an open interval containing a, except possibly at a. $f(x)$ has the limit L as x approaches a, or $\lim_{x\to a} f(x) = L$, means that for every real number $\varepsilon > 0$ there exists a $\delta > 0$, such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$.*

The condition "open interval" ensures that a is an interior point of an area where $f(x)$ is defined (except possibly at a itself). It allows x to approach a from any direction. The inequality $|f(x) - L| < \varepsilon$ indicates that the distance between $f(x)$ and L is less than ε, which can be as small as desired. The inequality $0 < |x-a| < \delta$ denotes a condition that x differs from a and is located in a's "neighborhood," which consists of all numbers with a distance from a less than δ. Why not use $|x-a|<\delta$ instead? This would strengthen the definition's condition because it requires that $|f(x)-L|<\varepsilon$ be valid when $x=a$. In fact, we consider which number $f(x)$ approaches as x approaches a, rather than $f(a)$, which is not even necessary defined.

Geometrically, a function $y=f(x)$ can be represented in the xy-plane. We present the following definition to help you better understand the concept of the limit.

Definition 1.2.2. *A box $B((a,b),\delta,\varepsilon)$, where $\delta,\varepsilon>0$, is a rectangle bordered by lines $x=a-\delta$, $x=a+\delta$, $y=b-\varepsilon$, $y=b+\varepsilon$. $B((a,b),\delta,\varepsilon)$ is also called a bound box for the function $f(x)$ if all points $(x,f(x))$ are located within the interior region of the rectangle whenever $0<|x-a|<\delta$.*

The formal definition 1.2.1 then can be interpolated as: $\lim_{x\to a} f(x)=L$ means that for every $\varepsilon>0$, there is a bound box $B((a,L),\delta,\varepsilon)$ for f for some $\delta>0$.

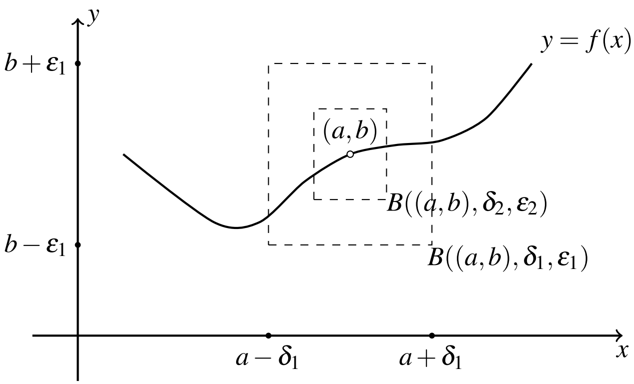


Figure 1.2: Bound boxes of $f(x)$

Lemma 1.2.3. *Let $f(x)$ be a real function. The following conditions are equivalent.*

(1) $\lim_{x\to a} f(x)=L$*;*

(2) There is an infinite sequence of bound boxes for $f(x)$:

$$B\left((a,L),\delta_1,1\right),B\left((a,L),\delta_2,\frac{1}{2}\right),\cdots,B\left((a,L),\delta_m,\frac{1}{m}\right),\cdots$$

The proof of this Lemma is similar to that of Lemma 1.1.3, and we will leave as an exercise for the reader.

As an example, we will show that some functions do not meet the criterion in Lemma 1.2.3 at a certain point, and thus the limit of the function does not exist at this point.

Example 1.2.1.

$$\lim_{x\to 0}\sin\frac{1}{x} \text{ does not exist}$$

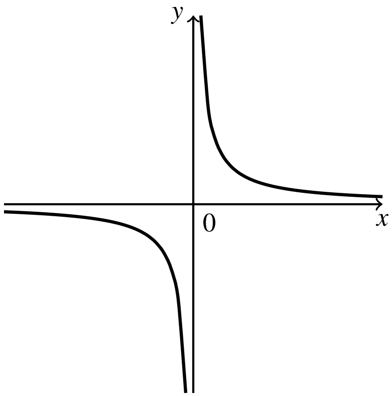


Figure 1.3: $y=\frac{1}{x}$

We will first examine this function graphically. To see more clearly, we begin with the graph of $y=\dfrac{1}{x}$ (Figure 1.3). When x approaches 0 from the right, $y\to\infty$; when x approaches 0 from the left, $y\to-\infty$. Assume x move from 1 to 0. Then $1/x$ will rapidly grow and covers all real numbers greater than 1. This causes the sine function's value oscillates between -1 and 1 indefinitely.

Figure 1.4 shows that the magnitude of y is not squeezed as x approaches 0. That is, if ε is small enough ($\varepsilon<1$ in this case), no box $B((a,b),\delta,\varepsilon)$ can hold the curve $y=\sin\frac{1}{x}$ where $x\in(-\delta,\delta)$. This is a visual intuition.

Formally, we will show that there exists a $\varepsilon>0$ such that for any $\delta>0$, we can always find a x such that $0<|x-0|<\delta$ and

$$\left|\sin\frac{1}{x}-L\right|>\varepsilon \tag{1.5}$$

Note that if $x=\dfrac{2}{(2n+1)\pi}$, where n is any natural number, then $\dfrac{1}{x}=$

$n\pi + \frac{\pi}{2}$ and $\sin\frac{1}{x}$ is either -1 or 1, depending on whether n is odd or even. Suppose δ is chosen, to find such x, let

$$x = \frac{2}{(2n+1)\pi} < \delta$$

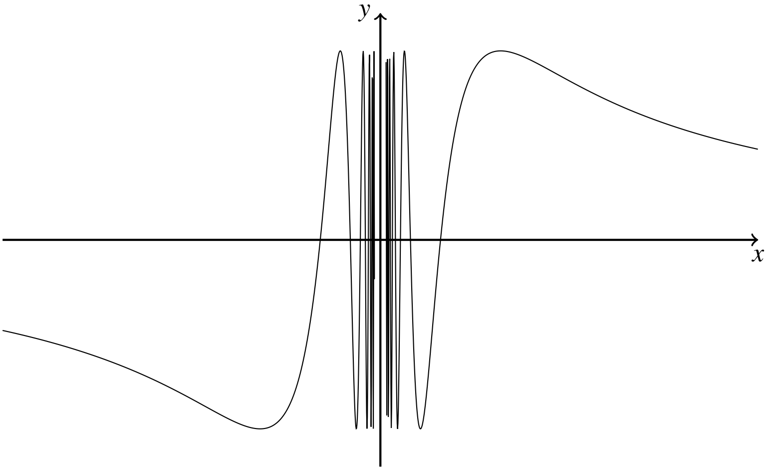


Figure 1.4: $y = \sin\frac{1}{x}$

This is equivalent to $n > \frac{1}{\delta\pi} - \frac{1}{2}$. Now we can make same argument as in Example 1.1.2, and conclude that there exists x that satisfies $0 < |x-0| < \delta$ and the inequality (1.5).

When x approaches a number a, it can approach a from the left or the right. The outcome may vary. Thus, the concept of the one-sided limit must be introduced. The one-sided limit has the same formal definition as the limit, except instead of $0 < |x-a| < \delta$, we write $0 < x-a < \delta$ (right-handed) or $0 < a-x < \delta$ (left-handed.). Writing $x \to a^-$ and $x \to a^+$ for "x approaches a from the left" and "x approaches a from the right," respectively. It is clear that $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$. Consider the following function:

$$f(x) = \begin{cases} x & x \leq a \\ x+1 & x > a \end{cases}$$

This is a pair of linear functions. The function is discontinuous at a. The (one-side) limit exists if x approaches a from one side, but the limits differ in different directions. Namely,

$$\lim_{x \to a^-} f(x) = a$$

$$\lim_{x \to a^+} f(x) = a + 1$$

Even two one-sided limits are equal, that is, if the limit exists at this point, the limit is not necessarily the same as the function's value. For instance,

$$f(x) = \begin{cases} 1 & x \neq a \\ 0 & x = a \end{cases} \tag{1.6}$$

Obviously, $\lim_{x \to a} f(x) = 1$ but $f(a) = 0$. This example demonstrates that even if a function is "disconnected" at one point, the function's limit can still exist. In other words, the existence of the limit is not sufficient for continuity.

Definition 1.2.4. [Continuity] *The function $f(x)$ is said to be* continuous *at a if*

1. $\lim_{x \to a} f(x) = L$ *for some real number L;*
2. $f(a) = L$

We will now summarize the relationship between the continuity and the existence of a function's limit at a given point.

$$f(x) \text{ is continuous at } a \Leftrightarrow \begin{cases} \lim\limits_{x \to a} f(x) \text{ exists} \Leftrightarrow \begin{cases} \lim\limits_{x \to a^-} f(x) \text{ exists} \\ \lim\limits_{x \to a^+} f(x) \text{ exists} \\ \lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) \end{cases} \\ \lim\limits_{x \to a} f(x) = f(a) \end{cases}$$

Consider the function

$$f(x) = x \sin \frac{1}{x}$$

Figure (1.5) show the graph. Since the magnitude of a Sine function is bounded by 1, the function approaches 0 as x approaches 0. To prove it formally, that is, for each $\varepsilon > 0$, we need to find a $\delta > 0$, such that $\left|x \sin \frac{1}{x}\right| < \varepsilon$ whenever $0 < |x| < \delta$. In this case, for a given ε, we simply choose $\delta = \varepsilon$, then

$$\left|x \sin \frac{1}{x}\right| \le |x| \left|\sin \frac{1}{x}\right| \le |x| < \delta = \varepsilon$$

This demonstrates the existence of $\lim_{x\to 0} f(x)$. However f is not continuous at 0 since f is not defined at 0.

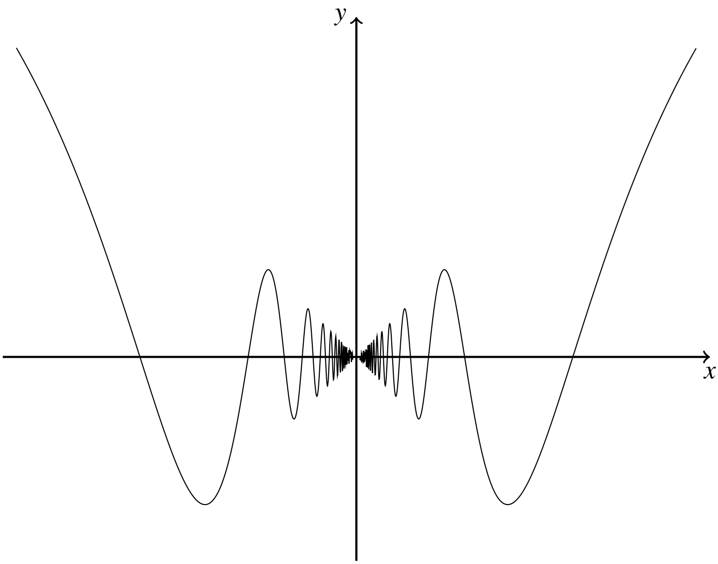


Figure 1.5: $y = x \sin \frac{1}{x}$

There are many limit theorems, but we won't go through all of them here. Instead, let's focus on the "Squeeze Theorem", which we can use later.

Imagine a bottle floating in the sea. The height of the bottle changes as the waves come and go, but it always remains somewhere between the crest and the trough of the waves. Now, suppose that when the ocean becomes calm, the height of the crest and trough approach each other, as does the height of the bottle. This idea can help us understand the squeeze theorem.

Theorem 1.2.5 (Squeeze Theorem)**.** *Let f,g,and h be functions defined on some interval except at a, and*

$$g(x) \leq f(x) \leq h(x)$$

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$$

then $\lim_{x\to a} f(x) = L$

Proof. By assumption, there is $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \implies g(x) \leq f(x) \leq h(x)$$

Given $\varepsilon > 0$, there are $\delta_2, \delta_3 > 0$, such that

$$0 < |x-a| < \delta_2 \implies |g(x) - L| < \varepsilon \implies g(x) - L > -\varepsilon$$

$$0 < |x-a| < \delta_3 \implies |h(x) - L| < \varepsilon \implies h(x) - L < \varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, we have

$$0 < |x-a| < \delta \implies -\varepsilon < g(x) - L \leq f(x) - L \leq h(x) - L < \varepsilon$$

This implies that

$$0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$$

□

The first condition, $g(x) \leq f(x) \leq h(x)$, can be relaxed by stating that "$f(x)$ is bounded by $g(x)$ and $h(x)$. " In other words, we allow $g(x) > h(x)$ but $f(x)$ must remain between the two. Take $f(x) = x\sin\frac{1}{x}$ once more. Since the Sine function is bounded by 1 and -1, $f(x)$ is bounded by $g(x) = x$ and $h(x) = -x$. As x approaches zero, so do $g(x)$ and $h(x)$. Applying the squeeze theorem, we have

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0 \tag{1.7}$$

Again, we must remember that $f(x)$ is not defined at point 0.

Let's work on the following limit problem as yet another helpful illustration:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \tag{1.8}$$

This appears to be a quotient of two functions at first. The relative "speed" of x and $\sin x$ determines where $f(x)$ goes. Both approach 0 as x approaches zero, although the limit is not as obvious (Figure 1.6).

Draw a unit circle and an angle x, where $0 < x < \frac{\pi}{2}$ as in Figure (1.6).
$OC = OA = 1$,
$BC = OC \cdot \sin x = \sin x$,
$AD = OA \cdot \tan x = \tan x$,
The area of $\triangle COA = \frac{1}{2}\sin x$
The area of $\triangle DOA = \frac{1}{2}\tan x$
The area of a unit circle is π,
The area of sector OAC is

$$\pi \cdot \frac{x}{2\pi} = \frac{x}{2},$$

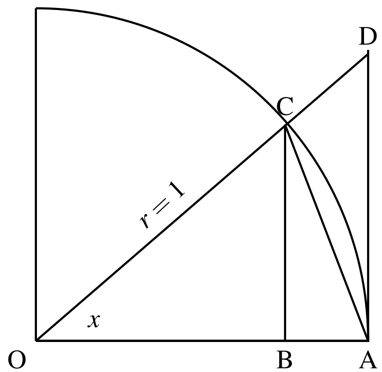


Figure 1.6: $\sin x \leq x \leq \tan x$

Since

$$\text{Area}(\triangle COA) \leq \text{Area(sector } OAC) \leq \text{Area}(\triangle DOA),$$

we have

$$\frac{1}{2}\sin x \leq \frac{x}{2} \leq \frac{1}{2}\tan x \tag{1.9}$$

Multiplying each term by $\frac{2}{\sin x}$ yields

$$1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$$

Taking reciprocals, we get

$$1 \geq \frac{\sin x}{x} \geq \cos x \tag{1.10}$$

We begin with the assumption $0 < x < \frac{\pi}{2}$, while the inequality (1.10) is also valid for $x < 0$ (in 4th quadrant). In this case ($x < 0$), only the direction of inequality (1.9) needs to be changed. Noting that the left and the right side of equation (1.10) approach to same limit: $\lim 1 = 1$, and $\lim_{x\to 0}\cos x = 1$, and applying Squeeze Theorem, we complete the proof of (1.8).

Exercises

1. Define
$$f(x) = x^2 \cos \frac{1}{x}$$
Show that $\lim_{x\to 0} f(x) = 0$. Is $f(x)$ continuous at 0?

2. Define
$$g(x) = \begin{cases} x^2 \cos \frac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0; \end{cases}$$
Is $g(x)$ continuous at 0 ?

3. Prove that the following function is continuous nowhere except at 0:
$$f(x) = \begin{cases} x & \text{if } x \text{ is a rational number;} \\ -x & \text{otherwise ;} \end{cases}$$

4. Find
$$\lim_{x\to 0} \frac{\tan^2 3x}{x \sin 5x}$$

5. Applying the Squeeze Theorem, calculate
$$\lim_{x\to\infty} \frac{3x + \sin x}{x + \cos x}$$

6. Assume $\lim_{x\to 1} f(x)$ exists and
$$\frac{x^2 - x + 5}{x + 2} \leq \frac{f(x)}{x - 3} \leq \frac{2x^2 + x + 2}{x + 2}$$
Find $\lim_{x\to 1} f(x)$.

1.3 Find Limits or Extrema: Examples

Depending on the type of functions we are working with, there are different ways to find the limit. A non-constant linear function, for instance, has no limit as the independent variable goes to infinite. Some functions, like

$$e^{-x}, \quad \frac{1}{x} \quad \text{and } \arctan(x) \ (x \geq 0),$$

each has one-side limit as $x \to \infty$. If a function is formed as $f(x)/g(x)$, and both $f(x) \to \infty,\ g(x) \to \infty$, or both $f(x) \to 0,\ g(x) \to 0$, it is said to have $\frac{\infty}{\infty}$ or $\frac{0}{0}$ *indeterminate* forms, respectively. Other indeterminate forms include $\infty - \infty$, 1^∞, ∞^0, and 0^0. In Calculus, we have more tools to find limits or extrema. Here, we use some straightforward methods.

A quadratic function has the form

$$f(x) = ax^2 + bx + c \quad (a \neq 0)$$

Depending on the sign of a, it is either unlimited upward or downward. On the other hand, it is bounded in the opposite direction, which is used in many applications. We will look why this occurs:

$$\begin{aligned}
f(x) &= ax^2 + bx + c \\
&= a\left(x^2 + \frac{b}{a}x\right) + c \\
&= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}
\end{aligned}$$

Since $(x + b/2a)^2 \geq 0$, it is clear that $-(b^2 - 4ac)/4a$ is the minimum value of $f(x)$ if $a > 0$, or the maximum value if $a < 0$. To be more precise,

$$f\left(-\frac{b}{2a}\right) = -\frac{b^2 - 4ac}{4a} \tag{1.11}$$

Consider the following real-life example: A boy standing on the bank of a river throws a baseball to the other side of the river. Assume the baseball's initial speed is $v = 20m/s$ and it has a 45° angle to the ground. For the sake of simplicity, disregard the boy's height and air resistance, and assume the gravitational acceleration of $g = 10m/s^2$. What is the maximum river width through which the baseball can pass? (Figure 1.7)