Singular Equations of Waves and Vibrations

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Contents

1 Introduction **1**

2 Waves and Vibrations in One Dimension **5**
2.1 Harmonic oscillator 5
2.2 Wave equation in one dimension 9
2.3 Singular equation. Regularization 14
2.4 Vibrations in one dimension 19

3 Waves and Vibrations in Two Dimensions **27**
3.1 Two-dimensional waves 27
3.2 Weyl-Sommerfeld integrals 29
3.3 Singular equation. Regularization 30
3.4 Poisson equation 32
3.5 Elastic waves in two dimensions 32
3.6 Vibrations in two dimensions. Bessel functions 37
3.7 Singular source. Helmholtz equation 40

4 Waves and Vibrations in Three Dimensions **45**
4.1 Three-dimensional waves 45
4.2 Regularization 47
4.3 Vibrations in three dimensions. Spherical harmonics . 49
4.4 Vibrations in three dimensions. Spherical Bessel functions 52
4.5 Vibrations. Singular source at the centre 54
4.6 Eigenfunctions and eigenvalues 57
4.7 Surface "waves" 58
4.8 Connection with two dimensions and one dimension . 60
4.9 Vibrations of a half-space 62
4.10 Navier-Cauchy equation for a half-space. Tensorial force 65
4.11 Vibrations of a sphere 69

5 Cylindrical and Conical Waves **73**
5.1 Weyl-Sommerfeld expansion. Waves 73
5.2 Waves and vibrations 77
5.3 Weyl-Sommerfeld expansion. Vibrations 80
5.4 Weyl-Sommerfeld expansion in two dimensions 82
5.5 Seismology 84

6 Vibrations of an Elastic Half-Space **87**
6.1 Introduction 87
6.2 Vibration equation 89
6.3 Surface-wave contribution 95
6.4 Concluding remarks 98
6.5 A particular case 99

7 Pulses and Scattering **103**
7.1 Propagation of pulses 103
7.2 Rayleigh flute 105
7.3 Bethe aperture 111
7.4 Sommerfeld edge 113
7.5 Loaded vibrations 115
7.6 Ringing of the bells 119
7.7 Collapse of a tower 123
7.8 Electromagnetic field on the edge 126

8 Fluids **129**
8.1 Fluids 129
8.2 Waves in fluids. Tsunami 132
8.2.1 Primary waves 133
8.2.2 Interaction with the surface 134
8.2.3 Secondary waves 138
8.3 Vibrations 142
8.3.1 Half-space 142
8.3.2 Two superposed fluids 147
8.3.3 Plane-parallel surfaces 148
8.4 Storm breakers 150
8.4.1 Oil on the surface 151
8.4.2 The effect of gravity 154

9 Seismic Waves **161**
9.1 Elastic wave equation 161
9.2 Coulomb potential regularization 163
9.3 Seismic wave regularization 165

10 Main Shock Equation **169**
10.1 Secondary waves 169
10.2 Main shock 172
10.3 Cherenkov equation 174
10.3.1 The case $v < c$ 175
10.3.2 The case $v > c$ 177
10.3.3 Transition radiation 178

11 Appendix **181**
11.1 Improper expressions 181
11.2 Bessel functions 184
11.3 The delta function 184
11.4 Singular equation for harmonic oscillator 186

Index **189**

1 Introduction

Let us consider a continuous medium, consisting of small, identical domains. The domains are so small that we can attach to each a continuous position $\boldsymbol{r}$. At the same time, the domains are sufficiently large that we can attach a mass and a mass density to each domain. Let us assume that a domain placed at $\boldsymbol{r}$ suffers a displacement $\boldsymbol{u}(t, \boldsymbol{r})$ at the moment of time t, and, therefore, has a velocity $\boldsymbol{v} = \frac{d\boldsymbol{u}}{dt}$. Then, Newton's equation of motion for each domain reads

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{f} \ , \tag{1.1}$$

where $\boldsymbol{f}$ is a force per unit mass. This equation is invariant under Galilei transformations.

The total time derivative is $\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v} grad$, such that equation (1.1) can also be written as

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} grad \boldsymbol{v} = \boldsymbol{f} \ . \tag{1.2}$$

If the force $\boldsymbol{f}$ arises from pressure variations, this is Euler's equation for the ideal fluid. If $\boldsymbol{v}$ is sufficiently small, or varies slowly with distance, we may neglect the quadratic term in velocities in equation (1.2) (the transport term), and get

$$\frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{f} \ . \tag{1.3}$$

This equation is not invariant anymore under Galilei transformations. The force $\boldsymbol{f}$ includes two contributions: an internal force $\boldsymbol{f}_i$, and an external force $\boldsymbol{f}_e$. The internal contribution is related to second-order spatial derivatives of the displacement. For instance, for sound (compression and dilation) in fluids or solids the internal force $\boldsymbol{f}_i$ is proportional to $grad\, div \boldsymbol{u}$, for shear displacement in solids $\boldsymbol{f}_i$ is proportional to $-curl\, curl\boldsymbol{u}$. In some cases the internal force can be expressed with the laplacian. In these cases, we write equation (1.3) (within the same approximation of neglecting the transport term) as the (standard) wave equation

$$\frac{\partial^2 \boldsymbol{u}}{\partial^2 t^2} - c^2 \Delta \boldsymbol{u} = \boldsymbol{f}_e \ , \qquad (1.4)$$

where c has the dimensions of a velocity. The approximation of neglecting the transport term requires $\mid \partial \boldsymbol{u}/\partial t \mid \ll c$. In addition, if the domains are made up of particles, u should be much smaller than the mean inter-particle (relative) distance, for the consistency of the continuous-medium description and the thermal equilibrium.[1]

As said above, the wave equation is not invariant under Galilei transformations. However, as it is well known, it is invariant under Lorentz transformation. For a velocity V along the x-coordinate the Lorentz transformations are

$$x' = \alpha x + c\beta t \ , \ t' = \tfrac{1}{c}\beta x + \alpha t \ ,$$

$$\alpha = \tfrac{1}{\sqrt{1-V^2/c^2}} \ , \ \beta = \tfrac{V/c}{\sqrt{1-V^2/c^2}} \ , \qquad (1.5)$$

$$\alpha^2 - \beta^2 = 1 \ ,$$

and the invariance of the wave equation $\partial^2 u/\partial t^2 - c^2 \partial^2 u/\partial x^2 = f_e$ can be immediately verified. On the other hand, for velocities V much smaller than c ($V \ll c$), which is precisely the condition of validity of the standard wave equation, the Lorentz transformations (equations (1.5)) are reduced to the Galilei transformations, and the wave equation becomes invariant under Galilei transformations; the wave equation (1.4) becomes now equivalent to equation (1.2).

For $V \to \pm c$ the Lorentz transformations give $x = \pm ct$ ($dx = \pm cdt$) and the standard wave equation becomes $0 = f_e$, which is meaningless. The equations of electromagnetism imply both a velocity $\boldsymbol{v}$ of a continuous medium and a universal velocity c (the speed of

[1] M. Apostol, *Physical Kinetics*, Cambridge Scholars Publishing, Newcastle upon Tyne (2020); see also "Non-inertial electromagnetic effects in matter. Gyromagnetic effect", Solid State Commun. **152** 1567 (2012).

light in vacuum). They lead to standard wave equations without the condition $v \ll c$. In fact, the velocity $\boldsymbol{v}$ does not appear explicitly in electromagnetic equations.[2] This means that a medium (aether) is superfluous for electromagnetism. However, the invariance under Lorentz transformations requires all the velocities be smaller than the universal velocity c, such that the electromagnetic equations become meaningless for v greater than c.

A special class of wave equations have a singular (external) force, *i.e.* a force proportional to $\delta(t)$, $\delta(\boldsymbol{r})$ or derivatives of the δ-functions. For $\delta(t)\delta(\boldsymbol{r})$ these equations provide the Green functions which help in constructing particular solutions of wave equations. At the same time, they may be viewed as singular equations, which may raise an intrinsic interest. Besides being discontinuous (non-analytic) or even singular in some cases, their solutions may include superfluous contributions of the free equations, which should be removed. They need a regularization procedure. The regularization procedure is presented in this book, and applied, especially, to the two- and three-dimensional Navier-Cauchy equations with a tensorial singular force. This tensorial force is the force acting in seismic foci.[3]

The distinction is emphasized in this book between waves and vibrations. The waves are governed by Cauchy initial conditions and the causality principle, while the vibrations are governed by boundary conditions for indefinite times. A particular example is provided by vibrations of a string and a circular cymbal, and the occurrence of resonances is underlined. Some other applications to pulses, elastic solids and fluids are presented, as well as a method of including the function boundary values as sources. The elastic vibrations of a homogeneous and isotropic half-space are presented by introducing new plane-wave vector functions. Singular equations for the seismic main shock and the Cherenkov radiation are also discussed. The book presents direct and efficient methods of solving some wave equations, both for waves and vibrations.

[2] M. Apostol, *Statistical Physics*, Cambridge Scholars Publishing, Newcastle upon Tyne (2021).

[3] B. F. Apostol, *The Theory of Earthquakes*, Cambridge International Science Publishing, Cambridge (2017); *Introduction to the Theory of Earthquakes*, Cambridge International Science Publishing, Cambridge (2017); *Seismology*, Nova, NY (2020).

2 Waves and Vibrations in One Dimension

2.1 Harmonic oscillator

Let us consider the well-known equation of a harmonic oscillator

$$\ddot{u} + \omega_0^2 u = S(t) \ , \tag{2.1}$$

where the unknown (real) function $u(t)$ is the coordinate of the oscillator, ω_0 is its eigenfrequency, the known function $S(t)$ is the source and t denotes the time. Our problem is to determine the unknown function $u(t)$. First, we need to specify the domain of the variable t for the function $u(t)$ (any function must have a domain of definition). We choose the domain $0 < t < \infty$. It is worth noting that the point $t = 0$ is not included, such that the derivative is defined. Of course, the source function $S(t)$ should be defined over the same domain. Second, we note that if we give the function u and its derivative at $t = 0$, the equation provides the second-order derivative at $t = 0$, so we may construct the solution as an analytic series at $t = 0$, providing $S(t)$ is analytic; we can continue the process for any point. It follows, in these conditions, that the solution exists, and is unique and analytic. This is known as the Cauchy-Kovalewskaya theorem; the initial ($t = 0$) conditions (which may also be called "boundary" conditions for the initial time) are known as defining the Cauchy problem for our equation. We write our initial conditions as

$$u(t = 0) = u_0 \ , \ \dot{u}(t = 0) = u_1 \ , \tag{2.2}$$

where the functions at $t = 0$ are understood as the limits for $t \to 0^+$. The solution is the sum of the free solution u_f, *i..e.* the solution of the homogeneous equation

$$\ddot{u}_f + \omega_0^2 u_f = 0 \ , \tag{2.3}$$

and a particular solution u_p of equation (2.1): $u = u_f + u_p$. The free solution is $u_f = A\cos\omega_0 t + B\sin\omega_0 t$, where the coefficients A and B are determined by the initial conditions. We get the solution

$$u(t) = [u_0 - u_p(0)]\cos\omega_0 t + \frac{1}{\omega_0}[u_1 - \dot{u}_p(0)]\sin\omega_0 t + u_p(t) \ . \quad (2.4)$$

It remains to determine a particular solution u_p.

The most direct method of finding a particular solution of equation (2.1) is to use the solution G of the equation

$$\ddot{G} + \omega_0^2 G = \delta(t) \ , \quad (2.5)$$

where $\delta(t)$ is the Dirac delta-function. Then, the particular solution is

$$u_p(t) = \int_0 dt' G(t-t')S(t') \ ; \quad (2.6)$$

indeed, if we apply the operator $\frac{d^2}{dt^2} + \omega_0^2$ to this equation, we get immediately $\ddot{u}_p + \omega_0^2 u_p = S(t)$. It remains to give a consistent and acceptable sense to equations (2.5) and (2.6).

The solution G of equation (2.5) is called the Green function of the equation of the harmonic oscillator. Equation (2.5) may be viewed as the equation of a harmonic oscillator with a "concentrated" source $\delta(t)$. It is a very meaningful equation from the physical standpoint. It is very different from the problem formulated for equation (2.1) above.

Indeed, first, we notice that $\delta(t)$ in equation (2.5) requires the extension of the domain to negative t, at least in a neighbourhood of $t = 0$. Second, $\delta(t)$ is not analytic, so we may not expect an analytic G. Third, initial conditions at $t = 0$ are meaningless for equation (2.5), because $\delta(t)$ is undetermined at $t = 0$. Such functions, with discontinuities, non-analytic and, in general, not determined, may have a sense only as generalized functions (distributions).[1]

Of course, G in equation (2.5) is defined up to free solutions. We can leave them aside, especially as we are interested in a particular solution u_p given by G. On the other hand, we can extend the domain

[1] V. S. Vladimirov, *Equations of Mathematical Physics*, Dekker, NY (1971).

of definition of equation (2.1) to $-\infty < t < +\infty$ with $u = 0$ and $S = 0$ for all $t < 0$. This amounts to multiplying equation (2.1) by the step function $\theta(t)$, leading to

$$\frac{d^2}{dt^2}(\theta u) + \omega_0^2(\theta u) = \theta S + u_0\dot{\delta}(t) + u_1\delta(t) \ . \tag{2.7}$$

Then we can re-define the source as

$$\widetilde{S}(t) = \theta(t)S(t) + u_0\dot{\delta}(t) + u_1\delta(t) \tag{2.8}$$

and get the solution by using the Green function (equation (2.6)). This way, we can include the initial conditions for equation (2.1) in the solution given by the Green function.[2] This is the one-dimensional form of the Green theorem for solution. We note that the step function is not determined for $t = 0$ (it is a distribution). The θ-multiplication procedure was used for treating electrical polarization in semi-infinite media (half-spaces), scattering by rough surfaces and the penetration depth of an electric field in a semi-infinite classical plasma.[3]

The above observation of a vanishing solution for negative times is extremely important. It may serve as a means of determining the Green function. According to equation (2.5), a pulse source appears at $t = 0$, so we are interested in the behaviour of the oscillator at the subsequent times $t > 0$, while the oscillator is at rest for all $t < 0$. This is the principle of causality. Its application gives retarded solutions: the source has effects only in the future, not backwards in time, in the past. The effect of a cause comes from the past (retarded solution), not from the future (advanced solution). Noteworthy, we avoid carefully the moment $t = 0$, which is, from the physics standpoint, a very sound procedure: at that moment the source acts, there cannot be a meaningful "solution" G. It seems that the Green functions and the generalized equations look more physical than the classical (analytic) functions and equations. Our world does indeed involve discontinuities (non-analyticity).

[2] M. Apostol, *Equations of Mathematical Physics*, Cambridge Scholars Publishing, Newcastle upon Tyne (2018).

[3] M. Apostol, *Essays in Electromagnetism and Matter*, Lambert, Saarbruecken (2013) and References therein; "Penetration depth of an electric field in a semi-infinite classical plasma", Optik **220** 165009 (2020).

Moreover, the causality principle imposes certain restrictions upon the way of deriving the solution. Indeed, it is easy to see that the free solution u_f is not vanishing for $t < 0$; similarly, the particular solution u_p given by equation (2.6) may not be vanishing for $t < 0$. On the other hand, it is easy to see, by comparing equation (2.4) to the solution given by the source $\widetilde{S}$, that the free solution corresponds to the $u_{0,1}$-contributions to $\widetilde{S}$, so we need $u_p(0) = 0$ and $\dot{u}_p(0) = 0$. As we will see shortly, these conditions are fulfilled for the harmonic oscillator.

By extending the domain to the whole real axis we can use the Fourier representation. In fact, solution (2.6) is the Fourier representation of equation (2.1). Unfortunately, the Fourier transform of equation (2.5),

$$G(t) = -\frac{1}{2\pi}\int d\omega \frac{e^{-i\omega t}}{\omega^2 - \omega_0^2} \ , \tag{2.9}$$

is an improper integral. We need to give a sense to this integral. This is done by using the principle of causality. Indeed, for $t < 0$ we should perform the integration in the upper half-plane, and should get zero. Therefore, the poles $\pm\omega_0$ should lie in the lower half-plane, which will give a non-zero result for $t > 0$. Consequently, ω should be replaced by $\omega + i\varepsilon$, $\varepsilon \to 0^+$, in equation (2.9),

$$\begin{aligned} G(t) &= -\tfrac{1}{2\pi}\int d\omega \tfrac{e^{-i\omega t}}{(\omega+i\varepsilon)^2-\omega_0^2} = \\ &= -\tfrac{1}{2\pi}\int d\omega \tfrac{e^{-i\omega t}}{\omega^2-\omega_0^2+isgn(\omega)\varepsilon} \ , \end{aligned} \tag{2.10}$$

or

$$G(t) = -\frac{1}{2\pi}\int d\omega \frac{e^{-i\omega t}}{(\omega - \omega_0 + i\varepsilon)(\omega + \omega_0 + i\varepsilon)} \ , \tag{2.11}$$

so we get

$$G(t) = \theta(t)\frac{\sin\omega_0 t}{\omega_0} \ . \tag{2.12}$$

We can check that this G satisfies equation (2.5). The causality principle determines the Green function. $G(0) = 0$ (in the sense $G(0^-) = 0$, $G(0^+) = 0$) and $\dot{G}(t) = \theta(t)\cos\omega_0 t$ (undetermined for $t = 0$). It is noteworthy that the Green function and the particular solution are determined by the eigenfrequency ω_0 of the oscillator, which governs the free solutions. The causality prescription of encircling the poles $\omega = \pm\omega_0$ indicates that the source is viewed as a "boundary condition" in the ω-variable.

According to equation (2.6), the particular solution is

$$u_p(t) = \int_0^t dt' \frac{\sin\omega_0(t-t')}{\omega_0} S(t') \ , \tag{2.13}$$

$u_p(0) = 0$, $\dot{u}_p(0) = 0$ and, from equation (2.4), the solution is

$$\begin{aligned} u(t) = u_0 \cos\omega_0 t + \tfrac{1}{\omega_0} u_1 \sin\omega_0 t + \\ + \textstyle\int_0^t dt' \frac{\sin\omega_0(t-t')}{\omega_0} S(t') \ , \ t > 0 \ . \end{aligned} \tag{2.14}$$

The same solution is obtained by using the source given by equation (2.8):

$$\begin{aligned} \theta(t)u(t) = \theta(t)u_0 \cos\omega_0 t + \theta(t)\tfrac{1}{\omega_0} u_1 \sin\omega_0 t + \\ + \theta(t) \textstyle\int_0^t dt' \frac{\sin\omega_0(t-t')}{\omega_0} S(t') \ . \end{aligned} \tag{2.15}$$

Equations (2.14) and (2.15) are known as the d'Alembert representation of the solution of the harmonic-oscillator equation. A damping term can be included in equation (2.1). This term produces a damped oscillation which fades out in the transient regime; thereafter, the stationary solution given above settles down.

2.2 Wave equation in one dimension

We write the wave equation in one dimension as

$$\frac{1}{c^2}\ddot{u} - u'' = S(t,x) \ , \tag{2.16}$$

where c is the wave velocity. The time is denoted by t and the spatial coordinate is denoted by x. We denote by upper dots the time derivatives and by upper slashes the spatial derivatives. A choice is to view the x-dependence of the solution $u(t,x)$ as the restriction to some x-domain. We note that if we give the function and its derivative at the initial moment, the second-order spatial derivative at that moment is determined. Then, we are left with a second-order differential equation in the variable t, which requires two initial conditions to be determined, for instance $u(0,x)$ and $\dot{u}(0,x)$. This is the Cauchy problem for our equation. The one-dimensional wave equation (2.16) can be viewed as describing compression and dilation in a thin rod, or transverse local displacement of a string.[4]

We may perform a spatial Fourier transform of equation (2.16),

$$\frac{1}{c^2}\ddot{u}(t,q) + q^2 u(t,q) = S(t,q) \ , \tag{2.17}$$

and write the solution as

$$u(t,q) = A\cos cqt + B\sin cqt + u_p(t,q) \ , \tag{2.18}$$

where the A, B-contribution is the free solution u_f and u_p is a particular solution. The reverse q-Fourier transform shows that the free solution is a sum of two functions depending on $x \pm ct$. We can see indeed that we have two coefficients A and B, which can be determined by the initial conditions. We can write the free solution as

$$u_f(t,x) = A(x-ct) + B(x+ct) \ , \tag{2.19}$$

where A and B are arbitrary functions. The particular solution is given by

$$u_p(t,x) = \int dt' \int dx' G(t-t', x-x') S(t', x') \ , \tag{2.20}$$

where the Green function is the solution of the equation

$$\frac{1}{c^2}\ddot{G} - G'' = \delta(t)\delta(x) \ . \tag{2.21}$$

[4] A. N. Tikhonov and A. A. Samarskii, *Equations of Mathematical Physics*, Dover, NY (1963); P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, NY (1953); L. Landau and E. Lifshitz, *Course of Theoretical Physics*, vol. 7 (*Theory of Elasticity*), Elsevier, Oxford (1986).

From the initial conditions $u(0,x)$ and $\dot{u}(0,x)$ we get

$$u(0,x) = A(x) + B(x) + u_p(0,x) \ , \qquad \dot{u}(0,x) = -cA'(x) + cB'(x) + \dot{u}_p(0,x) \ , \tag{2.22}$$

a system of equations which determine the functions A and B (up to a constant; the constant does not apear in $A+B$). This is known as the d'Alembert solution for the Cauchy problem in one dimension.

In order to solve equation (2.21) we take the time Fourier transform, with the same prescription regarding the integration, as given by the causality principle: we require $G=0$ for $t<0$. This requirement gives us retarded waves. Equation (2.21) becomes

$$G''(\omega,x) + \frac{\omega^2}{c^2}G(\omega,x) = -\delta(x) \ . \tag{2.23}$$

This is the equation of the harmonic oscillator (equation (2.5)), with interchanged variables t, x and ω_0 replaced by ω/c. As we will see shortly, the boundary values for x are dictated by the conditions imposed upon t (variable ω). The formal solution is given by

$$G(\omega,x) = \frac{1}{2\pi}\int dq \frac{e^{iqx}}{q^2-\omega^2/c^2} \ . \tag{2.24}$$

According to the causality principle this integral should be viewed as

$$G(\omega,x) = \frac{1}{2\pi}\int dq \frac{e^{iqx}}{(q-\omega/c-i\varepsilon)(q+\omega/c+i\varepsilon)} \tag{2.25}$$

($\omega \to \omega + i\varepsilon$, $\varepsilon \to 0^+$). We can see that the integration over q is dictated by the ω-poles. For $x>0$ we must integrate over the upper half-plane; for $x<0$ we integrate over the lower half-plane. Therefore, we get

$$G(\omega,x) = \frac{ic}{2\omega}e^{i\omega|x|/c} \ , \tag{2.26}$$

or, more exactly,

$$G(\omega,x) = \frac{ic}{2\omega+i\varepsilon}e^{i\omega|x|/c} \ . \tag{2.27}$$

This Green function is distinct from the Green function of the oscillator given by equation (2.12). Taking the inverse Fourier transform we get

$$G(t,x)=\frac{ic}{4\pi}\int d\omega\frac{e^{-i\omega(t-|x|/c)}}{\omega+i\varepsilon}\ . \tag{2.28}$$

According to the causality principle, for $t-\mid x\mid/c<0$ the integration in the upper half-plane is zero (the pole is below), while for $t-\mid x\mid/c>0$ the integration in the lower half-plane gives $c/2$. Therefore,

$$G(t,x)=\frac{c}{2}\theta(t-\mid x\mid/c)\ . \tag{2.29}$$

It is worth noting that this equation depends on $x\pm ct$ ($\neq 0$), as the free solution does.

The same result can be obtained if we perform first the integration over ω in the Fourier transform of equation (2.23),

$$G(t,q)=\int d\omega\frac{e^{-i\omega t}}{q^2-(\omega+i\varepsilon)^2/c^2}=\theta(t)c\frac{\sin cqt}{q} \tag{2.30}$$

(again, the poles are $\omega=\pm cq$, as for free waves). The inverse Fourier transform of this function,

$$G(t,x)=\frac{c\theta(t)}{4\pi i}\int dq\frac{e^{iq(x+ct)}-e^{iq(x-ct)}}{q} \tag{2.31}$$

has the pole $q=0$ on the real axis; it is reduced to the integral

$$\int dq\frac{\sin pq}{q}=\pi sgn(p)\ ,\ \left(\int dq\frac{\cos pq}{q}=0\right)\ ; \tag{2.32}$$

we get again equation (2.29).

The Green function given by equation (2.29) is a step-wise wave with two wavefronts $\mid x\mid=ct$, which propagate with velocities $\pm c$, ongoing from $t=0$ (a two-walls wave). The particular solution of the wave equation (2.16) (equation (2.20)) satisfies the causality principle. The free solution does not fulfill the causality principle. We may restrict ourselves to the domain $t>0$, $x>0$. We note that the causality-principle condition leads to a non-vanishing solution for

$x < 0$. Therefore, we can obtain the solution by extending it to the whole real t-axis and multiplying by $\theta(t)$; but we cannot do the same thing for the variable x.

Doing so, equation (2.16) becomes

$$\begin{gathered}\frac{1}{c^2}\frac{\partial^2}{\partial t^2}[\theta(t)u] - \frac{\partial^2}{\partial x^2}[\theta(t)u] = \\ = \theta(t)S(t,x) + \frac{1}{c^2}\dot{\delta}(t)u(0,x) + \frac{1}{c^2}\delta(t)\dot{u}(0,x) \ ;\end{gathered} \tag{2.33}$$

by making use of the Green function, we get

$$\begin{gathered}\theta(t)u(t,x) = \\ = \frac{1}{2}\theta(t)\left[\theta(x-ct)u(0,x-ct) + \theta(x+ct)u(0,x+ct)\right] + \\ + \frac{1}{2c}\theta(t)\theta(x-ct)\int_{x-ct}^{x+ct} dx'\dot{u}(0,x') + \\ + \frac{1}{2c}\theta(t)\theta(ct-x)\int_{0}^{x+ct} dx'\dot{u}(0,x') + \\ + \frac{c}{2}\int_0 dt' \int_0 dx' \theta(t-t'- \mid x-x' \mid /c)S(t',x') +\end{gathered} \tag{2.34}$$

This is the d'Alembert solution. We note that it satisfies the causality principle ($= 0$ for $t < 0$) and $u(0,x) = u(0,x)$, $\dot{u}(0,x) = \dot{u}(0,x)$, but it has non-vanishing values for $x < 0$; also, this solution and its spatial derivative have well-defined values for $x = 0$. We may restrict ourselves to $x > 0$. Boundary conditions (like conditions for $x = 0$) are impossible for the wave equation which satisfies the causality principle. The solution obtained above is valid for any x, and we may take its restriction to whatever domain we choose.

It is worth noting that both the Green function and the particular solution are constructed by the poles $\omega = \pm cq$ corresponding to the free solution. The source is treated as a "boundary condition" for the free solution. Since the free solution depends only on two variables $x \pm ct$, there exist only two coefficients to be determined by two (independent) conditions, in agreement to the second order of the equation.

2.3 Singular equation. Regularization

Equation (2.21) can also be viewed as a wave equation, not necessarily an equation for a Green function; it is rather an unphysical equation, since the total force $\delta(t)\int dx\delta(x)$ is not zero. In order to do this we need to place the source in the domain $t > 0$, $x > 0$. Therefore, we get the equation

$$\frac{1}{c^2}\ddot{u} - u'' = \delta(t)\delta(x - x_0) \ , \tag{2.35}$$

where $t = 0$ is understood as $t = 0^+$ and $x_0 > 0$. We call it a singular equation. Obviously, the particular solution of this equation is

$$u_p(t, x) = \frac{c}{2}\theta(t- \mid x - x_0 \mid /c) \tag{2.36}$$

(a two-walls wave). Equation (2.35) and its particular solution given by equation (2.36) have a noteworthy particularity: for all times and positions, except the wavefronts $\mid x - x_0 \mid= ct$, the solution is a constant (0 or $c/2$). The presence of the constant $c/2$ leads to an unphysical situation, because the solution is determined up to an arbitrary constant. The only physically relevant solutions should exist at the discontinuous wavefronts, where, however, the solution is undetermined. The restriction to these points is the regularization of the solution. We denote the regularized solution by u. The θ function occurs because we require an analytic solution, except for some points. We use equation (2.29) as a Green function, since we need analytic functions for the particular solutions of the wave equation with sources; but equation (2.29) is not a meaningful solution of the singular wave equation (2.35).

The particular solution given by equation (2.36) satisfies the causality principle ($u_p = 0$ for $t < 0$). We may view it as the restriction of u_p to $x > 0$. Its initial and boundary values are

$$\begin{gathered} u_p(0, x) = 0 \ , \ \dot{u}_p(0, x) = \tfrac{c^2}{2}\delta(x - x_0) \ , \\ u_p(t, 0) = \tfrac{c}{2}\theta(t - x_0/c) \ , \ u_p^{'}(t, 0) = \tfrac{1}{2}\delta(t - x_0/c) \ . \end{gathered} \tag{2.37}$$

We can include an initial condition $u(0, x)$, according to equation (2.34), since $u_p(0, x) = 0$. But we cannot include $\dot{u}(0, x)$, since

$\dot{u}_p(0,x) \neq 0$. Also, boundary conditions are impossible. We can see that a concentrated source $\delta(x-x_0)$ renders impossible the (full) Cauchy problem of the wave equation. The waves given by equation (2.36) start to propagate from the source, to the right and to the left, at the initial moment $t=0$. The wave propagating to the left reaches the point $x=0$ after the time x_0/c and gives a constant "displacement" $c/2$ to this point for all the subsequent times, while the wave propagating to the right continues its way to infinity. We note that the above functions (equations (2.37)) are not defined for $x=x_0$, $t=0$ and $x=0$, $t=x_0/c$, which are particular wavefronts, *i.e.* points which satisfy the equation $|x-x_0|=ct$; this equation (of the wavefronts) defines the "characteristics" of the wave equation. Indeed, for $x-x_0=\pm ct$ the wave equation (2.35) becomes the meaningless equality $0=\frac{1}{c}\delta^2(t)$.

Under the above conditions the singular equation (2.35) may be accepted as a meaningful wave equation, though its regularized solution u is reduced to a point.

The story of the singular equation (2.35) is not yet finished. Let us first assume that the end point $x=0$ is fixed. It follows that an opposite displacement $u(t,0)=-\frac{c}{2}\theta(t-x_0/c)$ should appear at this end, in order to have a total zero displacement. For subsequent times this displacement becomes $-\frac{c}{2}\theta(t-x/c-x_0/c)$; this is a solution of the singular wave equation which propagates to the right with velocity c. It originates in the mirror-image source placed at $-x_0$. Therefore, the wave is reflected (with change of sign) by the fixed end. The superposition of the incident wave and the reflected wave gives a vanishing displacement. Of course, this reflected solution should be regularized. This reflected solution may be viewed as a scattered wave.

The wave acts upon the end point $x=0$ by a force $c^2u''\mid_{x=0}=\frac{c}{2}\delta'(t-x_0/c)$. Indeed, the internal force (per unit mass) in equation (2.35) is c^2u''. We note that we can use the solution u_p for calculating the derivatives of the regularized solution u. The internal force c^2u'' acts upon each point $x>0$, on behalf of its neighbouring points. It is a "volume" force. The force $c^2u''\mid_{x=0}$ acts upon an internal point x where $x\to 0^+$. This is the internal point placed at $x=0$. This internal force is distinct from the "surface" force. Indeed, the internal force arises from "surface" forces $s(x)$ acting upon the "surface" of any domain; the internal force is given by

$$s(x + \Delta x/2) - s((x - \Delta x/2) = c^2 u''(x) \cdot \Delta x \ , \tag{2.38}$$

where Δx is of the order of the dimension of the domains; hence we can see that the "surface" force is $s = c^2 u'(x)$. Since the domains are contiguous, the "surface" forces acting upon them are vanishing; except the "surface" force acting at the end point $x = 0$, which is $s = c^2 u'(0)$. The "surface" force acting at the end point $x = 0$ may, at most, generate a displacement of the whole medium, but it cannot generate a local motion. It may serve as a boundary condition, but not as a force generating motion.

Therefore, the incident wave acts by a force $c^2 u'' \mid_{x=0} = \frac{c}{2}\delta'(t - x_0/c)$ upon the end point $x = 0$. Since the end is fixed, it reacts back to the medium by a force $-\frac{c}{2}\delta'(t - x_0/c)$ placed at $x = 0$. We may imagine that a "surface" force is present at the end point, for a short time, which, divided by a small extension, of the order of the dimension of the domains, cancels out the incident force, such that the total force at the end point is zero. According to the Huygens principle, the original wave ceases its existence at this point and a new wave appears, generated by this reaction force. The Huygens principle acts at any point in the medium, only that the relevant part of the force generated by the incident wave at any internal point is identical to the incident force, such that the motion is transmitted through the medium.

Usually, the reflection is treated for a continuous presence of the incident wave at the surface, which is a vibration problem (as we shall see shortly); in that case, a force may have a continuous presence at the surface.

The localization of the end-point force at $x = 0$ raises problems in determining its form. The solution of this indetermination resides in the physical meaning of the δ function. We should use $l\delta(x)$ for the spatial localization of this force, where l is the small extension of the function $\delta(x)$. Consequently, we can represent the force as $-\frac{c}{2}\delta'(t - x_0/c)l\delta(x)$. However, this force is placed at $x = 0$, so half of it is lost outside the domain $x > 0$. Therefore, we should multiply this result by 2, such that the force is

$$-c\delta'(t - x_0/c)l\delta(x) \ . \tag{2.39}$$

The cutoff l allows the representation $-\frac{c^2}{l}\delta(t-x_0/c)$ for $-c\delta'(t-x_0/c)$, which, together with $l\delta(x)$, eliminates the cutoff parameter l. Consequently, we are lead to the wave equation

$$\frac{1}{c^2}\ddot{u}-u''=-\delta(t-x_0/c)\delta(x) \ . \tag{2.40}$$

The solution of this equation is

$$u(t,x)=-\frac{c}{2}\theta(t-x/c-x_0/c) \tag{2.41}$$

for $x>0$, *i.e.* precisely the reflected wave found above. Of course, the solution should be regularized.

If the end point $x=0$ is "free" the force acting upon it on behalf of the incident wave sets it in motion. Since the medium is consistent, *i.e.* it cannot be disrupted, an equal and opposite force should appear, exactly as for a fixed end point. Therefore, the situation is completely similar to that described above.

The representation given by equation (2.39) may also be cast as

$$-c^2 l\delta(t)\delta'(x) \tag{2.42}$$

(where we measure the time from $t=x_0/c$); this leads to the equation

$$\frac{1}{c^2}\ddot{u}-u''=-l\delta(t)\delta'(x) \ . \tag{2.43}$$

We put this source at $x=x_0$, leave aside the cutoff l and write the equation as

$$\frac{1}{c^2}\ddot{u}-u''=\delta(t)\delta'(x-x_0) \ . \tag{2.44}$$

This representation is preferable, since the total force acting upon the body in this case is zero, which is a physical situation ($\int dx\delta'(x-x_0)=0$). The particular solution of this equation can be obtained from the solution of equation (2.35) by differentiation with respect to x:

$$u_p(t,x)=-\frac{1}{2}sgn(x-x_0)\delta(t-\mid x-x_0\mid/c) \tag{2.45}$$

(a pulse wave). In contrast to the wave equation (2.35), this wave equation in one dimension has meaningful solutions, in the sense that they are already regularized. The solution u_p is the regularized solution u. We note that on the wavefronts $x-x_0=\pm ct$ equation (2.44) becomes $0=\frac{1}{c}\delta(t)\delta'(t)$, which is a valid equality.
Its initial and boundary values are

$$\begin{array}{c}
u(0,x)=-\frac{c}{2}sgn(x-x_0)\delta(x-x_0)\ , \\
\\
\dot{u}(0,x)=-\frac{c^2}{2}sgn(x-x_0)\delta'(x-x_0)\ , \\
\\
u(t,0)=\frac{1}{2}\delta(t-x_0/c)\ ,\ u^{'}(t,0)=\frac{1}{2c}\delta'(t-x_0/c)\ .
\end{array}\tag{2.46}$$

We can see that the Cauchy problem becomes meaningless for this equation ($u(0,x)\neq 0$, $\dot{u}(0,x)\neq 0$). This is expected for a solution whose support is reduced to points ($\delta(t-\mid x-x_0\mid/c)$ is different from zero for $x=x_0\pm ct$).[5] We may view $u(t,x)$ as the full solution of equation (2.44), with the restriction $x>0$ (it satisfies the causality principle, $u=0$ for $t<0$). This solution describes two concentrated pulses, with opposite signs, propagating to the right and to the left with velocities $\pm c$ from $x=x_0$ and for all times $t>0$. Once arrived at $x=0$, after time x_0/c, the wave produces a reaction force $-c^2u''\mid_{x=0}$, which, following the procedure described above leads to the equation

$$\frac{1}{c^2}\ddot{u}-u''=-\delta(t-x_0/c)\delta'(x)\ ,\tag{2.47}$$

with solution

$$u(t,x)=\frac{1}{2}\delta(t-x/c)\tag{2.48}$$

for $x>0$. This wave propagates to the right, as a pulse, with velocity c. It is a reflected wave. If we have a finite-size body, with x restricted to $0<x<l$, the pulse waves will suffer multiple reflections on both ends.

Singular equations in one dimension with sources $\delta(t)\delta(x)$, $\delta(t)\delta'(x)$ describe the propagation of electric pulse on a metallic wire (the physical problem corresponds to the latter).[6]

[5] B. F. Apostol, "On the Lamb problem: forced vibrations in a homogeneous and isotropic elastic half-space", Arch. Appl. Mech. **90** 2335 (2020).
[6] M. Apostol, "Electric pulse on a metallic wire", Optik **204** 164095 (2020).

The equation of elastic waves in a three-dimensional solid with a tensorial force localized at $\boldsymbol{R}_0$ is

$$\ddot{\boldsymbol{u}} - c_t^2 \Delta u_i - (c_l^2 - c_t^2) grad\, div \boldsymbol{u} = \boldsymbol{F} \ , \tag{2.49}$$

where $\boldsymbol{u}$ is the displacement, $c_{l,t}$ are the longitudinal and transverse velocities and the force (per unit mass) is given by

$$F_i = m_{ij} T \delta(t) \partial_j \delta(\boldsymbol{R} - \boldsymbol{R}_0) \ , \tag{2.50}$$

m_{ij} being the (symmetric) tensor of the moment per unit mass and T denotes the duration of the force; the cartesian indices are $i,\, j = x, y, z$.[7] In one dimension, along the coordinate x, with $\boldsymbol{R}_0 = (x_0, 0, 0)$, this equation becomes

$$\begin{aligned} \ddot{u}_x - c_l^2 u_x^{''} &= m_{xx} T \delta(t) \delta'(x - x_0) \ , \\ \ddot{u}_\alpha - c_t^2 u_\alpha^{''} &= m_{\alpha x} T \delta(t) \delta'(x - x_0) \ , \end{aligned} \tag{2.51}$$

where $\alpha = y,\, z$. These equations are of the form given by equation (2.44). The source is distributed uniformly in a plane perpendicular to the x-axis.

2.4 Vibrations in one dimension

The solution $u(t, x)$ of the wave equation

$$\frac{1}{c^2}\ddot{u} - u'' = S(t, x) \tag{2.52}$$

can also be viewed as the restriction to some t-domain, or the whole temporal space. Therefore, we may take the time Fourier transform of this equation,

$$u''(\omega, x) + \frac{\omega^2}{c^2} u(\omega, x) = -S(\omega, x) \ , \tag{2.53}$$

[7] B. F. Apostol, *The Theory of Earthquakes*, Cambridge International Science Publishing, Cambridge (2017); *Introduction to the Theory of Earthquakes*, Cambridge International Science Publishing, Cambridge (2017); *Seismology*, Nova, NY (2020).

and write the solution as

$$u(\omega, x) = A\cos\frac{\omega x}{c} + B\sin\frac{\omega x}{c} + u_p(\omega, x) \ , \tag{2.54}$$

where the A, B-contribution is the free solution u_f and u_p is a particular solution. The coefficients A and B are determined by two conditions. These are boundary conditions for the x-variable. For instance, we can give $u(t,0)$ (a condition known as the Dirichlet condition) and $u'(t,0)$ (a von Neumann condition) for the boundary $x=0$. As shown before $c^2u'(t,0)$ is the "surface" force acting upon the end point $x=0$; the condition $u'(t,0)=0$ means a free end point $x=0$. The condition $u(t,0)=0$ means a fixed end point. A combination $\alpha u(t,0)+\beta u'(t,0)=0$, with α, β some coefficients, may represent a certain motion of the end point. Insufficient conditions lead to a free coefficient. By the reverse ω-Fourier transform, we can see that the free solution depends on $t \pm x/c$. Therefore, the free solution can be written as a sum of two functions

$$u_f(t, x) = A(t - x/c) + B(t + x/c) \ . \tag{2.55}$$

The particular solution is given by

$$u_p(t, x) = \int dt' \int dx' G(t-t', x-x')S(t', x') \ , \tag{2.56}$$

where the Green function G is given by the equation

$$\frac{1}{c^2}\ddot{G} - G'' = \delta(t)\delta(x) \ . \tag{2.57}$$

The functions A and B are determined by the boundary conditions

$$\begin{aligned} A(t) + B(t) + u_p(t,0) &= u(t,0) \ , \\ -\tfrac{1}{c}A'(t) + \tfrac{1}{c}B'(t) + u_p^{'}(t,0) &= u'(t,0) \ . \end{aligned} \tag{2.58}$$

We note that the time may be extended from $-\infty$ to $+\infty$, or we may take the restriction of the solution to any time domain. In a vibration problem the time is indefinite and the boundary conditions act over the entire time domain.

In order to solve equation (2.57) for the Green function we may assume $G = 0$ for $x < 0$, the space domain being $0 < x < +\infty$. Therefore, in the spatial Fourier transform

$$G(t,x) = \frac{1}{2\pi}\int dqG(t,q)e^{iqx} \tag{2.59}$$

we place the q-poles in the upper half-plane; indeed, for $x < 0$ the integration in the lower half-plane gives then zero, while for $x > 0$ the integration in the upper half-plane gives a non-zero result. Consequently, we replace q by $q - i\varepsilon$, $\varepsilon \to 0^+$ in $G(t,q)$. We note that this procedure is similar to a "causality" principle for x.

Let us take the spatial Fourier transform of equation (2.57):

$$\frac{1}{c^2}\ddot{G}(t,q) + q^2G(t,q) = \delta(t) \ . \tag{2.60}$$

The solution of this equation is

$$\begin{gathered} G(t,q) = -\tfrac{c^2}{2\pi}\int d\omega \tfrac{e^{-i\omega t}}{\omega^2 - c^2q^2} = \\ = -\tfrac{c^2}{2\pi}\int d\omega \tfrac{e^{-i\omega t}}{(\omega - cq + i\varepsilon)(\omega + cq - i\varepsilon)} = \tfrac{ic}{2}\tfrac{e^{-icq|t|}}{q - i\varepsilon} \ , \end{gathered} \tag{2.61}$$

such that the Green function is

$$G(t,x) = \frac{ic}{4\pi}\int dq \frac{e^{iq(x-c|t|)}}{q - i\varepsilon} = -\frac{c}{2}\theta(x - c \mid t \mid) \ . \tag{2.62}$$

We can see again that the Green function and the particular solution are given by the poles $\omega = \pm cq$, corresponding to the free solution. We note the similarity between the Green functions of waves (equation (2.29)) and vibrations (equation (2.62)), though the difference $\mid x \mid$ *vs* $\mid t \mid$ is essential. It shows that the waves propagate in the future in the whole space, while the vibrations oscillate at any position as a consequence of waves propagating both from the past and the future. The wave solution is causal, while the vibration solution is acausal.

Let us extend the variable x to $-\infty$ and multiply equation (2.52) by $\theta(x)$; the equation becomes

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\theta u) - \frac{\partial^2}{\partial x^2}(\theta u) = \theta S - \delta(x)u'(t,0) - \delta'(x)u(t,0) \ . \tag{2.63}$$

The solution of this equation is the d'Alembert solution

$$\begin{aligned}\theta(x)u(t,x) = \theta(x)\tfrac{1}{2}\left[u(t-x/c,0)+u(t+x/c,0)\right]+\\ +\theta(x)\tfrac{c}{2}\int_{t-x/c}^{t+x/c}dt'u'(t',0)- \qquad (2.64)\\ -\theta(x)\tfrac{c}{2}\int dt'\int_0 dx'\theta(x-x'-c\mid t-t'\mid)S(t',x')\ ,\end{aligned}$$

where we recognize the functions A and B introduced above.

Let us assume $u(t,0)=\sin\omega t$, $u'(t,0)=0$, $S(t,x)=0$, *i.e.* a harmonic oscillation of the end $x=0$ of a semi-infinite string; equation (2.64) gives the solution

$$u(t,x)=\sin\omega t\cos\omega x/c\ , \qquad (2.65)$$

i.e. the oscillating excitation of the boundary extends to the whole string; this is a stationary (standing) wave, arising from the superposition of two waves $\sin(\omega(t\pm x/c)$. We can see that one wave comes from the future. The separation of the dependence on the variables t and x is a vibration, *i.e.* a stationary (standing) wave, in contrast to the propagating waves which are travelling waves.

Let us assume $u(t,0)=\delta(t)$, $u'(t,0)=0$, $S(t,x)=0$, *i.e.* a pulse on the end. Equation (2.64) gives the solution

$$u(t,x)=\frac{1}{2}[\delta(t-x/c)+\delta(t+x/c)]\ , \qquad (2.66)$$

which describes the propagation along the string of the pulse generated at the end of the string, one pulse coming from the future. These are travelling waves. Actually, this is a propagating-wave problem, generated by $u(t,0)=\delta(t)$; it is not appropriate to treat it as a vibration problem, as shown by the unphysical solution we get. Similarly, a pulse source generates a travelling wall- or pulse-wave. We can see that vibrations are generated by the continuous presence of waves at the surface; pulses generate travelling waves.[8]

Let us consider a finite string, extending from $x=0$ to $x=l$. In order to implement the boundary conditions we may extend the domain to $-\infty<x<+\infty$ and multiply the function by $\theta(x)\theta(l-x)$.

[8]B. F. Apostol, "On the Lamb problem: forced vibrations in a homogeneous and isotropic elastic half-space", Arch. Appl. Mech. **90** 2335 (2020).