Mathematics in Industry

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CAMBRIDGE
SCHOLARS
PUBLISHING

Mathematics in Industry,
Edited by Angela Slavova

This book first published 2015

Cambridge Scholars Publishing

12 Back Chapman Street, Newcastle upon Tyne, NE6 2XX, UK

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

ISBN (10): 1-4438-6401-3, ISBN (13): 978-1-4438-6401-5

TABLE OF CONTENTS

Chapter Three: Chaotic Systems and their Application in Industry

Chapter Four: Industrial Applications in Mechanics

Chapter Five: Algorithms in Industrial Mathematics

Chapter Six: Network Applications in Industry

Chapter Seven: Nonlinear Waves and Simulations

Chapter Eight: Mathematical Physics Equations and Applications in Industry

Chapter Nine: Linear Algebra Applications

PREFACE

This book describes a wide range of problems concerning recent achievements in the field of industrial and applied mathematics. The main goal is to provide new ideas and research both for scientists, who develop and study mathematical methods and algorithms, and for researchers, who apply them to solving real-life problems. The book promotes basic research in mathematics leading to new methods and techniques useful to industry and science. This volume will be a medium for the exchange of information and ideas between mathematicians and other technical and scientific personnel.

The main topics are: Numerical Methods and Algorithms; Control Systems and Applications; Partial Differential Equations and Applications; Neurosciences (Neural Networks); Equations of Mathematical Physics, etc.

Many important real-life applications of partial differential equations and equations of mathematical physics are presented in the book (Chapters 1,7 and 8). More precisely, a non-local version of a nonlinear Schrödinger equation is studied, which is a theoretical description of wave propagation in PT-symmetric coupled wave guides and photonic crystals. Continuity of the solution map for the cubic 1D periodic nonlinear wave (NLW) equation is investigated. The Cauchy problem to the generalized sixth-order Boussinesq equation is studied. This problem arises in a number of mathematical models of physical processes, for example in the modeling of surface waves in shallow waters and in the dynamics of nonlinear lattices. A family of Modified Korteweg-de Vrez (MKDV) equations is delivered and is related to simple Lie algebra. G-strand equations are studied and peakon-antipeakon collisions are solved analytically and can be applied in the theory of image registration. The three-soliton interactions for the Manakov system are modeled by a perturbed complex Toda chain.

A survey is presented concerning chaotic systems and their application in industry (Chapter 3). A receptor-based Cellular Nonlinear Network model with hysteresis is studied. The dynamics and stability of this model are studied from the point of view of local activity theory and an edge-of-chaos domain is obtained. Continuous feedback control is applied in order to stabilize the system. The coupled FitzHugh-Nagumo neural system is studied in this survey, and stabilization of the discretized models is proposed which is simple to implement.

Industrial Applications in mechanics are presented (Chapter 4). Li-ion batteries are widely used currently in the automotive industry, in electronic devices, etc. Pole scale simulations are provided on 3D CT images of the porous electrodes.

Algorithms in industrial mathematics are investigated (Chapter 5). An improved algorithm for generating primitive Pythagorean triples is proposed, an algorithm based on a well-known construction by Barning and Hall. A similar construction is considered in the four-dimensional case of Pythagorean quadruples and the generalized case of relatively prime quadruples.

Another topic which is considered in the volume is network applications in industry (Chapter 6). A neural network for classification of plastic and non-plastic materials with blasting action after blow-up with coherent signals in the optical range is proposed. Another network application is a graphical user interface created to study the static equation of linear Cellular Neural Networks (CNN). An interactive web tool is developed to explore associations in networks built with Affymetrix transcriptional profiling data and other sources of genomics data.

Linear algebra applications are considered (Chapter 9). A general parametric AE-solution set is obtained, which appears in the domain of various industrial applications. This chapter also presents a review of the main results of the component-wise stability of Wang's parallel partition method for banded and tri-diagonal linear systems.

High performance and scientific topics are included in this volume (Chapter 2). The importance of the computing infrastructure is unquestionable for the development of modern science. This chapter presents an approach to the installation and configuration of a high performance computing infrastructure with grid access. The cluster comprises a large pool of computational blades and two powerful GPGPU-enabled servers. The Danish Eulerian Model is a powerful and sophisticated air pollution model. Novel developments in the up-to-date parallel implementation of the model are presented in this chapter. A field fire model is proposed which is based on game modeling using hexagonal cells and rules. A parallel version of the algorithm is run on the Blue Gene supercomputer.

This book has a very important role in the promotion of interdisciplinary collaboration between applied mathematics and science, engineering and technology.

I would like to thank very much Dr. Maya Markova for her help in preparing this volume.

Sofia, May 2014 Angela Slavova

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CHAPTER ONE

REAL-LIFE APPLICATIONS OF PDF

LOCAL WELL-POSEDNESS FOR THE CUBIC 1D PERIODIC SQUARE-ROOT KLEIN GORDON EQUATION

VLADIMIR GEORGIEV AND MIRKO TARULLI

Introduction

We consider the Cauchy problems associated with the following square-root KG equation

$$(i\,\partial_t - \sqrt{-\Delta + m^2})u = \sigma|u|^2 u \text{ for t} \geq 0, \quad (1)$$

where $m > 0$, $\sigma = \pm 1$, and $u(t,x)$ is 2π-periodic in x. If we have solutions $u(t,x) \in C([0,T]; H^s(0,2\pi))$, with $s > 1/2$, then the equation enjoys two conservation laws

$$\| u(t) \|_{L^2(0,2\pi)} = const$$

and

$$\frac{1}{2} \| (-\Delta + m^2)^{1/4} u(t) \|_{L^2}^2 + \frac{\sigma}{4} \| u(t) \|_{L^4}^4 = const. \quad (2)$$

Moreover we state:

Definition 1.1 *The problem (1) is well-posed in* $H^s(0,2\pi)$ *with* $s \in (0,1)$ *if for any* $R > 0$ *one can find* $T = T(R) > 0$ *such that for any initial data* $u(0) = f \in H^s$ *with* $\| f \|_{H^s} \leq R$ *one can define unique solution* $u(t,x) \in C([0,T]; H^s)$ *so that the solution map*

$$f \in B(R) = \{g \in H^s; \| g \|_{H^s} \leq R\} \to u(t,x) \in C([0,T]; H^s),$$

is continuous.

A stronger property is the uniform continuity of the solution map. In this direction we have our main result, that is

Theorem 1.2 *If one selects* $s \in (1/3,1/2)$, *then the Cauchy problem associated with*

$$(i\partial_t - \sqrt{-\Delta + m^2})u = |u|^2 u \; for \; t \geq 0, \tag{3}$$

cannot have a uniformly continuous solution map in H^s.

From now let us select $m = 1$ and indicate by $\sqrt{-\Delta + 1} = \langle D_x \rangle$. Then the above result is valid also for

$$(i\,\partial_t - \langle D_x \rangle)u = -|u|^2 u,$$

but in this case one expects some blow-up effect similar to the one obtained in [3]. To explain the idea of the proof, let us look at a solution of the form

$$u(t,x) = u(t,x;s,\varepsilon) = v_{\geq 0}(t,x;s,\varepsilon) + w_\varepsilon(t,x), \tag{4}$$

where $w(t,x) = w_\varepsilon(t,x)$ satisfies

$$(i\,\partial_t - |D_x|)w = (|v+w|^2(v+w) - P_{\geq 0}(|v|^2)v) + S(D_x)(v+w), \tag{5}$$

with the smoothing operator

$$S(D_x) = \langle D_x \rangle - |D_x|,$$

and with zero initial data. In addition we choose $v(t,x) = v_{\geq 0}(t,x)$, as the solution of the modified equation for the NLW (see [2] for more details)

$$i(\partial_t - \partial_x)v = P_{\geq 0}(v|v|^2),$$

where $P_{\geq 0}$ is the operator

$$P_{\geq 0}\left(\sum_{k\in\mathbb{Z}} \hat{f}(k)e^{ikx}\right) = \hat{f}(0) + \sum_{k=1}^{\infty} \hat{f}(k)e^{ikx}.$$

We shall construct a family of $v_{\geq 0}(t,x)$ defined for any positive ε and for $j = 0{,}1$ as follows

$$v_\varepsilon^{(j)}(t,x) = v_{\geq 0}(t,x;j,s,\varepsilon) = \frac{a_j e^{-i\alpha_j t}}{1-c_0 e^{i(t(1-\gamma_j)+x)}}, \tag{6}$$

where

$$c_0 = c_0(\varepsilon) = \sqrt{1-\varepsilon}, a_j = a_j(\varepsilon) = m_j \varepsilon^{s+1/2}, \quad 0 < s < \frac{1}{2}, \tag{7}$$

$$m_j = m_j(\varepsilon) = 1 + \frac{j}{|\log\varepsilon|}, \; j = 0{,}1, \tag{8}$$

and with

$$\gamma_j = \frac{c_0^2 a_j^2}{1-c_0^2}, \; \alpha_j - \gamma_j = \frac{a_j^2}{(1-c_0^2)^2}. \tag{9}$$

We choose $\varepsilon > 0$ small and use the fact that $v_\varepsilon^{(j)}(t,x) = v_{\geq 0}^{(j)}(t,x;s,\varepsilon)$ introduced in (6) could be used in the proof of the fact that the solution map is not uniformly continuous; that is the statement of Theorem 1.2. Note that the relations (7) and (9) imply

$$2\gamma = \gamma(\varepsilon) = m^2\varepsilon^{2s}(1+O(\varepsilon)),$$

$$\alpha = \alpha(\varepsilon) = \; m^2\varepsilon^{2s-1}(1+O(\varepsilon)). \tag{10}$$

Furthermore we can apply the argument of Section 5 in [1] (see also [5]) so we shall obtain the following estimates

$$\| v_\varepsilon^{(1)}(0,\cdot) - v_\varepsilon^{(2)}(0,\cdot) \|_{H^s(0,2\pi)} \leq C\frac{1}{|\log\varepsilon|}, \tag{11}$$

with $C > 0$ and consequently, for suitable interval of type

$$\{t \sim \varepsilon^{1-2s}|\log\varepsilon|\},$$

we have:

$$\| v_\varepsilon^{(1)}(t,\cdot) - v_\varepsilon^{(2)}(t,\cdot) \|_{H^s(0,2\pi)} \geq D > 0, \tag{12}$$

with some $D > 0$ independent of $\varepsilon > 0$. Our main goal is to construct functions $u_\varepsilon^{(1)}(t,x), u_\varepsilon^{(2)}(t,x)$ so that

• $u_\varepsilon^{(1)}(t,x), u_\varepsilon^{(2)}(t,x)$ are solutions of the square - root KG equation (3) of the form (4),

• they have slightly smoother than $H^{1/2}$ regularity, i.e.

$$u_\varepsilon^{(1)}, u_\varepsilon^{(2)} \in C([0,T(\varepsilon)]; H^{s_1}(0,2\pi)),$$

for suitable choices of $s_1 > 1/2,\ T(\varepsilon) > 0,$

• the functions $u_\varepsilon^{(1)}, u_\varepsilon^{(2)}$ satisfy the estimates (11) and (12) with some $s \in (1/3,1/2)$.

The inequalities (11) and (12) for these solutions will imply the conclusion of Theorem 1.2.

Solutions of 1D square - root NLKG system as perturbations of Szegö type solutions

For any function $a: \mathbb{Z} \to \mathbb{C}$ we set

$$a(D)f(x) = \sum_{k\in\mathbb{Z}} a(k)\hat{f}(k)e^{ikx},$$

where $\hat{f}(k)$ is the Fourier coefficient of f. In particular we have

$$|D|f(x) = \sum_{k\in\mathbb{Z}} |k|\hat{f}(k)e^{ikx}. \tag{13}$$

We have also the partition of unity

$$I = P_+ + P_0 + P_-, \tag{14}$$

where

$$P_+(k) = \begin{cases} 1, & if \;\; k \; > \; 0; \\ 0, & otherwise, \end{cases} \quad \mathrm{P}_-(k) = \begin{cases} 1, & if \;\; k < 0; \\ 0, & otherwise, \end{cases}$$

by this we obtain

$$|D| = DP_+ - DP_- = (P_+ - P_-)D. \tag{15}$$

We shall use also the operators

$$P_{\geq 0} = P_+ + P_0, \quad P_{\leq 0} = P_- + P_0.$$

Let use recall from [2] that if one just picks up $m = 0$, then the square root Klein Gordon equation (1) can be transformed into a system

$$2i(\partial_t - \partial_x)u_{\geq 0} = Q_{\geq 0}(u_{\geq 0}, u_-), \tag{16}$$

$$i(\partial_t + \partial_x)u_- = Q_-(u_{\geq 0}, u_-).$$

The system (16) is simplified essentially when $u_- = 0$ and becomes a simple scalar equation of type

$$i(\partial_t - \partial_x)v_{\geq 0} = P_{\geq 0}(|v_{\geq 0}|^2 v_{\geq 0})\mathrm{for} t \geq 0. \tag{17}$$

Lemma 2.1 *For any* $s \in (1/3,1/2)$ *one can find solutions*

$$v_\varepsilon(t,x) \in C(\mathbb{R}; H^s(0,2\pi)),$$

of (17) having the form (6), i.e.

$$v_\varepsilon(t,x) = v_{\geq 0}(t,x;s,\varepsilon),$$

such that $P_-(v_\varepsilon) = 0$.

Making the substitution

$$u = v + w, \quad v(t,x) = v_\varepsilon(t,x), \tag{18}$$

in (16) we arrive at the following equation

$$(i\,\partial_t - |D_x|)w = (w+v)^2\overline{(w+v)} - v^2\overline{v} + P_-(v^2\overline{v}) + S(D)(v+w), \tag{19}$$

where

$$(w+v)^2\overline{(w+v)} - v^2\overline{v} = 2wv\overline{v} + \overline{w}v^2 + w^2\overline{v} + 2w\overline{w}v + w^2\overline{w}. \tag{20}$$

It is important to classify all terms on the right hand side of (19). First of all we notice that the last term, because of the smoothing nature of the operator $S(D)$ (see for instance [4] and reference therein) fulfills

Lemma 2.2 *Assume* $f_\varepsilon \in C([0,1], H^s)$, *for some* ε, $s = \frac{1}{2} - \varepsilon$, $\varepsilon > 0$, *then one has*

$$\| S(D) f_\varepsilon \|_{H^{a+s}} = \| (\langle D_x \rangle - |D_x|) f \|_{H^{a+s}} \leq C \| f_\varepsilon \|_{H^s},$$

for all $a \in [0,1/2]$ and with $C > 0$ independent from ε.

The above lemma suggests that we can concentrate on the remaining terms. Then we have linear combinations of the following:

1. Term $w^2\overline{w}$ cubic in w.
2. Terms $w^2\overline{v}$ and $w\overline{w}v$ quadratic in w.
3. Terms $wv\overline{v}$ and $\overline{w}v^2$ linear in w.
4. Term of type $P_-(vv\overline{v}) = P_-(v|v|^2)$.

In this way we have

$$(i\,\partial_t - |D_x|)w = \sum_{j=1}^{3} Q_j(v,w) + P_-(vv\overline{v}), \tag{21}$$

where

$$3Q_1(v,w) = 2wv\overline{v} + \overline{w}v^2, \tag{22}$$

$$Q_2(v,w) = w^2\overline{v} + 2w\overline{w}v,$$

$$Q_3(w) = w^2\overline{w}.$$

We can rewrite (21) as

$$(i\,\partial_t - |D_x|)(w + w_0) = \sum_{j=1}^{3} Q_j(v,w), \tag{23}$$

where w_0 is a solution to the linear equation

$$(i\,\partial_t - |D_x|)w_0 = P_-(vv\overline{v}). \tag{24}$$

Since $v(t,x) = v_\varepsilon(t,x)$ is a family of smooth solutions, such that

$$\| v_\varepsilon \|_{C([0,1];\dot{H}^\sigma)} \leq C\varepsilon^{s-\sigma}, \quad \| v_\varepsilon \|_{C([0,1];L^2)} \leq C\varepsilon^{s}, \tag{25}$$

and

$$\| v_\varepsilon \|_{C([0,1];L^\infty)} \leq C\varepsilon^{s-1/2}$$

(see Proposition 5.1), we see that is seems difficult to derive an estimate of type

$$\| w_0 \|_{C([0,1];H^\sigma)} \leq C\varepsilon^{\theta} < \infty, \tag{26}$$

with some $\sigma > 1/2, \theta > 0$. However, we shall overcome this obstacle and establish (26) by using suitable modified Bourgain spaces and estimates for v leading to the smoothing property (26). Once the smoothing property (26) is verified, we can follow the approach based on semi-classical estimates and show that (23) has solution w satisfying better error estimates.

Semi-classical estimates for the solution

We refer for this Section to the paper [2]. Given any $\sigma > 1/2$ and any smooth function $w(t,x)$ consider the semiclassical energy

$$E_{\sigma,\varepsilon}(w(t)) = \varepsilon^{-1} \| w(t,\cdot)) \|_{L^2}^2 + \varepsilon^{2\sigma-1} \| w(t,\cdot) \|_{H^\sigma}^2, \qquad (27)$$

by this fact along the section we shall assume

$$E_{\sigma,\varepsilon}(w_0(t)) = O(\varepsilon^\theta),$$

for some $\varepsilon > 0, \theta > 0$. It is easy to compare this semiclassical norm with the standard Sobolev norm. This is given by the

Lemma 3.1 *For any* $s, 0 < s \leq \sigma$

$$\| f(x) \|_{H^s} \leq C\varepsilon^{1/2-s}\sqrt{E_{\sigma,\varepsilon}(f)}. \qquad (28)$$

The construction of Szegö type solutions $v(t,x) = v_\varepsilon(t,x)$, *described in the previous section together with (25), guarantees that*

$$E_{\sigma,\varepsilon}(v(t)) \leq C\varepsilon^{2s-1}. \qquad (29)$$

For the equation (23) we have the following semi-classical estimate

Lemma 3.2 *For any* $\sigma \geq 0$ *there exists a constant* $C > 0$ *so that for any* $T \in (0,1)$ *we have*

$$2 \sup_{0\leq t\leq T} \sqrt{E_{\sigma,\varepsilon}(w(t))} \leq C \sup_{0\leq t\leq T} \sqrt{E_{\sigma,\varepsilon}(w_0(t))} + \qquad (30)$$

$$+C\left(\sum_{j=1}^{3} \int_0^T \sqrt{E_{\sigma,\varepsilon}(Q_j(t))}\ dt\right).$$

Proof: We have the following estimates

$$\| w(t,\cdot) \|_{L^2} \leq C \| w_0(t,\cdot) \|_{L^2} + \sum_{j=1}^{3} \int_0^t \| Q_j(\tau,\cdot) \|_{L^2_x} d\tau,$$

$$\| w(t,\cdot) \|_{\dot{H}^\sigma} \leq C \| w_0(t,\cdot) \|_{\dot{H}^\sigma} + \sum_{j=1}^{3} \int_0^t \| Q_j(\tau,\cdot) \|_{\dot{H}^\sigma} \, d\tau.$$

From these estimates we get out (30).

Thus we need the following

Lemma 3.3 *For any $\sigma > 1/2$ one can find a constant $C = C(\sigma) > 0$ so that for any $t, 0 \leq t \leq 1$ we have*

$$\sqrt{E_{\sigma,\varepsilon}(Q_3(t))} \leq C\big(E_{\sigma,\varepsilon}(w(t))\big)^{3/2}. \qquad (31)$$

We have also

Lemma 3.4 *For any $\sigma > 1/2$ one can find a constant $C > 0$ so that for any $t, 0 \leq t \leq 1$ we have*

$$\sqrt{E_{\sigma,\varepsilon}(Q_2(t))} \leq C E_{\sigma,\varepsilon}(w(t)) \ \varepsilon^{s-1/2}. \qquad (32)$$

We quote Proposition 5.1 and we can write the following semi-classical estimate for the Szegö type solutions $v(t,x) = v_\varepsilon(t,x)$

$$\| v(t,\cdot) \|_{L^\infty} \leq C\varepsilon^{s-1/2}, \quad \sqrt{E_{\sigma,\varepsilon}(v(t))} \leq C\varepsilon^{s-1/2}. \qquad (33)$$

In a similar way we find

Lemma 3.5 *For any $\sigma > 1/2$ one can find a constant $C > 0$ so that for any $t, 0 \leq t \leq 1$ we have*

$$\sqrt{E_{\sigma,\varepsilon}(Q_1(t))} \leq C\sqrt{E_{\sigma,\varepsilon}(w(t))} \ \varepsilon^{2s-1}. \qquad (34)$$

The semi-classical estimate (30) shows that we can set

$$g(T) = \sup_{0 \leq t \leq T} \sqrt{E_{\sigma,\varepsilon}(w(t))}, \quad \varepsilon_1 = \varepsilon^{1-2s}, \ \varepsilon^\theta = \varepsilon_1^{\theta_1}$$

and derive the following estimate

$$g(T) \leq C\left(\varepsilon_1^{\theta_1} + \int_0^T g(t)^3 dt + \int_0^T g(t)^2 \frac{dt}{\sqrt{\varepsilon_1}} + \int_0^T g(t) \frac{dt}{\varepsilon_1}\right). \quad (35)$$

Lemma 3.6 *If $g(t)$ is a continuous non - negative function satisfying (35) with $\theta_1 \in (0,1/2)$, then one can find $\varepsilon_0 > 0$ so that for $0 < \varepsilon_1 < \varepsilon_0$ we have the inequality*

$$g(T) \leq 2C\varepsilon_1^{\theta_1/2} \quad (36)$$

for

$$0 \leq T \leq T^*(\varepsilon_1) = \frac{\theta_1 \varepsilon_1}{2} \ |\log\varepsilon_1|.$$

Proof of Theorem 1.2

In this section, following the spirit of the paper [2], we shall complete the proof of Theorem 1.2 provided that the smoothing estimate (26), with some $\sigma > 1/2, \theta > 0$, is satisfied. As already shown in Lemma 3.6 we have the estimate

$$g(t) = \sup_{0 \leq s \leq t} \sqrt{E_{\sigma,\varepsilon}(w_\varepsilon(s))} \leq 2C\varepsilon_1^{\theta_1/2} \quad (37)$$

for

$$\varepsilon_1 = \varepsilon^{1-2s}, \ \theta_1 = \frac{\theta}{1-2s} > 0$$

and for

$$0 \leq t \leq T^*(\varepsilon_1) = \frac{\theta_1 \varepsilon_1}{2} \ |\log\varepsilon_1|.$$

Note that the estimate (28) implies

$$\| w_\varepsilon(t) \|_{H^s} \leq C\varepsilon^{1/2-s}. \quad (38)$$

The substitution (18) yields that for any $\varepsilon \in (0,1/2)$ and for any $j = 0,1$

$$u_\varepsilon^{(j)}(t,x) = v_{\geq 0}^{(j)}(t,x;s,\varepsilon) + w_\varepsilon^{(j)}(t,x),$$

is a solution to the equation (3) (with $m = 1$), i.e. for $u = u_\varepsilon^{(j)}(t,x)$ it solves

$$(i\,\partial_t - < D_x >)u_\varepsilon^{(j)} = |u_\varepsilon^{(j)}|^2 u_\varepsilon^{(j)} \text{fort} \geq 0,$$

with initial data

$$u_\varepsilon^{(j)}(0,x) = \frac{a_j}{1-c_0 e^{ix}}, \tag{39}$$

such that

$$c_0 = c_0(\varepsilon) = \sqrt{1-\varepsilon}, a_j = a_j(\varepsilon) = \left(1 + \frac{j}{|\log\varepsilon|}\right)\varepsilon^{s+1/2}. \tag{40}$$

Using Lemma 5.2, it is easy to verify that

$$\| u_\varepsilon^{(1)}(0,\cdot) - u_\varepsilon^{(0)}(0,\cdot) \|_{H^s}^2 = \sum_{k\geq 0} \langle k\rangle^{2s} |a_1 - a_0|^2 (1-\varepsilon)^k \leq \frac{C}{|\log\varepsilon|^2}. \tag{41}$$

The estimates (33) show that

$$\| v(t) \|_{H^s} \leq C\varepsilon^{1/2-s}\sqrt{E_{\sigma,\varepsilon}(v(t))} \leq C_1. \tag{42}$$

Our principal target shall be to establish that for some $D > 0$ we have

$$\| v_{\geq 0}^{(1)}(t,x;s,\varepsilon) - v_{\geq 0}^{(0)}(t,x;s,\varepsilon)) \|_{H^s} \geq D, \tag{43}$$

for t in a suitable interval; the estimates (37), (38) guarantee that the term

$$v_\varepsilon^{(j)}(t,x) = v_{\geq 0}^{(j)}(t,x;s,\varepsilon),$$

is dominant in the representation $u_\varepsilon^{(j)} = v_\varepsilon^{(j)} + w_\varepsilon^{(j)}$ so the estimates (41) and (43) will complete the proof of Theorem 1.2. For this we have to verify (41) only. To do this we shall concentrate the proof on the estimate of a suitable Fourier coefficient in the Fourier expansion of

$$v_\varepsilon^{(j)}(t,x) = \frac{a_j(\varepsilon)e^{-i\alpha_j(\varepsilon)t}}{1-c_0(\varepsilon)e^{i(x+t-\gamma_j(\varepsilon)t)}},$$

where the parameters $\alpha_j(\varepsilon), \gamma_j(\varepsilon)$ satisfy the asymptotical expansions

$$2\gamma_j(\varepsilon) = \varepsilon^{2s}\left(1 + \frac{j}{|\log\varepsilon|} + o\left(\frac{1}{|\log\varepsilon|}\right)\right),$$

$$\alpha_j(\varepsilon) = \varepsilon^{2s-1}\left(1 + \frac{j}{|\log\varepsilon|} + o\left(\frac{1}{|\log\varepsilon|}\right)\right), \tag{44}$$

as $\varepsilon \searrow 0$. One has the following

Lemma 4.1 *For any $d_0 > 0$ one can find constants $D, d_1, d_2 > 0$ with $d_1 < d_2 < d_0$ so that for any $\varepsilon \in (0,1/2)$, any $j = 0,1$ and any integer $k, 0 \le k \le N(\varepsilon) = (log2)/\varepsilon$, the Fourier coefficient*

$$C_k^{(j)}(t,\varepsilon) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{a_0(\varepsilon)e^{-i\alpha_j(\varepsilon)t}}{1-c_0(\varepsilon)e^{i(x+t-\gamma_j(\varepsilon)t)}}\ e^{-ikx}dx, \tag{45}$$

satisfies the estimate

$$\left|C_k^{(1)}(t,\varepsilon) - C_k^{(0)}(t,\varepsilon)\right| \ge D(1-\varepsilon)^{k/2}\varepsilon^{s+1/2}, \tag{46}$$

for

$$d_1\varepsilon^{1-2s}|log\varepsilon| \le t \le d_2\varepsilon^{1-2s}|log\varepsilon|. \tag{47}$$

It is easy to compare the Fourier coefficients in (45) with the standard Fourier coefficients

$$\widehat{v_\varepsilon^{(j)}}(t,k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v_\varepsilon^{(j)}(t,x)\ e^{-ikx}dx = \frac{a_j}{a_0}C_k^{(j)}(t,\varepsilon).$$

From the fact that

$$a_j(\varepsilon) - a_0(\varepsilon) = \varepsilon^{s-1/2}\left(\frac{1}{|\log\varepsilon|} + o\left(\frac{1}{|\log\varepsilon|}\right)\right),$$

thus Lemma 4.1 gives the following

Lemma 4.2 *For any $d_0 > 0$ one can find constants $D, d_1, d_2 > 0$ with $d_1 < d_2 < d_0$ so that for any $\varepsilon \in (0,1/2)$, any $j = 0,1$ and any integer $k, 0 \le k \le N(\varepsilon) = (log2)/\varepsilon$ the Fourier coefficients*

$$\widehat{v_\varepsilon^{(j)}}(t,k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v_\varepsilon^{(j)}(t,x)\ e^{-ikx}dx, \tag{48}$$

satisfy the estimate

$$\left|\widehat{v_\varepsilon^{(1)}}(t,k) - \widehat{v_\varepsilon^{(0)}}(t,k)\right| \ge D(1-\varepsilon)^{k/2}\varepsilon^{s+1/2}, \tag{49}$$

for

$$d_1\varepsilon^{1-2s}|log\varepsilon| \le t \le d_2\varepsilon^{1-2s}|log\varepsilon|. \tag{50}$$

To complete the proof of (41) we use Lemma 4.2 and the relations

$$\| v_{\ge 0}^{(1)}(t,x;s,\varepsilon) - v_{\ge 0}^{(0)}(t,x;s,\varepsilon)) \|_{H^s}^2 \ge$$

$$\ge C\sum_{k=0}^{N(\varepsilon)}\left|\widehat{v_\varepsilon^{(1)}}(t,k) - \widehat{v_\varepsilon^{(0)}}(t,k)\right|^2 \langle k\rangle^{2s} \ge D\sum_{k=0}^{N(\varepsilon)} \varepsilon^{2s+1}(1-\varepsilon)^k\langle k\rangle^{2s},$$

combined with the following estimate, which implies by the way the optimality of the one in Lemma 5.2:

Lemma 4.3 *There is a positive constant $C > 0$ so that for any $d > 0$, and $\theta \in [0,1]$ and for any $\varepsilon \in (0,1/2)$ we have*

$$\sum_{k=0}^{d/\varepsilon} (1+k)^\theta(1-\varepsilon)^k \ge \frac{C}{\varepsilon^{1+\theta}}. \tag{51}$$